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Exploiting MUS Structure to Measure Inconsistency of Knowledge Bases

Said Jabbour[1] and **Lakhdar Sais**[1]

Abstract. Measuring inconsistency is recognized as an important research issue for quantifying and handling inconsistencies in knowledge bases. Several logic-based inconsistency measures have been proposed. Minimal unsatisfiable and maximal satisfiable subsets are at the heart of the syntactic measures, while semantic inconsistency measures are often based on some paraconsistent semantics. In order to design interesting measures faithful to human rationality, many properties have been introduced to reach this goal. In this paper, we propose a new property called sub-additivity allowing to push further the ability to reorder knowledge bases according to their inconsistency degree. After pointing out the limitations of several measures to satisfy the sub-additivity property, we present a new measure based on a fine exploitation of the internal structure of the knowledge base, namely the structure of its associated minimal unsatisfiable subsets. Then, we show how its computation can be formulated as a nonlinear optimization problem. Finally, we prove that the new measure satisfies all the required properties while highlighting its interesting features.

1 Introduction

Reasoning about inconsistent knowledge bases (KBs) is one of the fundamental topics attracting growing interest from the AI community. It aims to quantify the amount of inconsistency, useful for guiding inconsistency resolving [14, 7, 5]. Its interest has been highlighted in several domains including software specifications [3], belief merging [33], news reports [18], integrity constraints [11], and multi-agents systems [20, 21].

These last years, several inconsistency measures have been proposed. Some of them [18, 12, 19, 26], focus on minimal unsatisfiable subsets (MUSes), pointed out as the elementary unit circumscribing inconsistency. Dually, maximal satisfiable subsets are also exploited in this context. Indeed, the maximal satisfiable subsets can be derived from the set of minimal unsatisfiable subsets using minimal hitting sets of the set of MUSes. They represent the different possible ways of restoring consistency of an inconsistent knowledge base. Among these measures, $I_{MI}(K)$ stated in [19] is defined as the number of MUSes of a knowledge base K, while considering the knowledge base with a high number of MUSes as the most inconsistent one. Unfortunately, such approach considers the contribution of each MUS to the whole inconsistency as independent from its possible interaction with the remaining MUSes. Recent work focuses on MUSes interaction through their intersections and consider that the inconsistency should take into consideration such overlaps. This observation leads to a new active research issue with several new inconsistency measures [21, 23]. These measures have been motivated from the cooperative multi-agents perspective [23]. A classical problem of inconsistency assessment related to MUSes strucutre, can be represented in the following knowledge bases:

$K_1 = \{p, \wedge\neg p, q, \neg q\}$;
$K_2 = \{p, \wedge\neg p \wedge q, \neg q\}$

K_1 contains two disjoint MUSes while K_2 holds interconnected MUSes. The recurrent questions are: if K_2 has the same inconsistency as K_1? Otherwise, How the inconsistency degree of K_2 is far from the one of K_1?

Works related to MUSes dependencies have been reported in many studies of inconsistency handling. In [36], hitting set based inconsistency measure is introduced. In [4], the authors highlighted dependence relation between MUSes by showing that inconsistency resolving is MUSes structure dependent. To facilitate the description of justificatory structure, in [2] the authors introduce a graph-based framework for capturing and analyzing relationships between justifications in OWL ontologies.

In parallel, and in order to consolidate or to justify the proposed measures, several properties have been proposed to judge the rationality of the proposed measures. Such properties were analyzed and discussed by many authors (see for example [6]). Among the properties that attracted much more attention, we can cite the dominance property [19] requiring a non decreasing inconsistency value when a consistent formula of a knowledge base is replaced with one of its logical consequences. A weaker form has been characterized using prime implicates, a canonical logic based representation of knowledge bases [22].

Following this research trend, in this paper, we propose a new approach to measure inconsistency. First, a new property called sub-additivity is proposed. Based on a finer analysis of the connections between MUSes, it allows us to push further the possibility to reorder knowledge bases with respect to their inconsistency degree. Secondly, we exploit the relationships between MUSes to design a new inconsistency metric satisfying the new property and some of the well-known rationality properties.

The contributions of this paper can be summarized as follows:

1. We introduce a new property called sub-additivity, and we discuss its relevance to the problem of measuring inconsistency.
2. We point out the limitations of the previous inconsistency measures and we propose a new one exploiting the inner-structure of the knowledge base, namely the structure of the graph-based representation of the set of MUSes.
3. We provide different ways for computing or approximating the inconsistency value, while providing results on some knowledge

[1] CRIL, Univ. Artois & CNRS, F-62300 Lens, France, email: {jabbour, sais@cril.fr}

bases.

The rest of this paper is organized as follows. After some preliminary definitions and notations, we briefly recall different proposals for measuring inconsistency relevant to the present work. In Section 3, after motivating the limitation of ind-additivity, we introduce our new property called sub-additivity. In Section 4, we introduce a new measure exploiting dependencies between MUSes. Section 5, provides a discussion of the properties of the introduced measure. In Section 6, we formulate the computation of our new inconsistency measure as an optimization problem and provide approximation. Finally, we give related work in Section 7, before concluding and giving some directions for future work in Section 8.

2 Preliminaries

A propositional language $\mathcal{L}$ is built over a finite set of propositional symbols $\mathcal{P}$ using classical logical connectives. We use letters (possibly indexed) $p, q, r, \ldots$ to represent atoms in $\mathcal{P}$, and Greek letters α, $\beta \ldots$ to denote formulae. The special letter $\perp$ denotes falsity (contradiction), while $\top$ denotes validity.

A literal is an atom p or its negation $\neg p$. A clause C is a disjunction of literals $(p_1 \vee \ldots \vee p_n)$. A formula α in *conjunctive normal form* (CNF) is a conjunction of clauses. Let $Var(\alpha)$ denotes the set of variables occurring in α. A knowledge base (in short KB) K is a finite set of propositional formulae. For a set S, we denote by $|S|$ its cardinality. We say that K is *inconsistent* if $K \vdash \perp$, where $\vdash$ is the classical consequence relation.

Minimal unsatisfiable subsets are often used to analyze inconsistency. Formally, this concept is defined as follows:

Definition 1 (MUS)**.** *Let K be a knowledge base and $M \subseteq K$. M is a* Minimal Unsatisfiable (or Inconsistent) Subset *(MUS) of K iff $M \vdash \perp$ and $\forall M' \subsetneq M$, $M' \nvdash \perp$. The set of all minimal inconsistent subsets of K is denoted $MUSes(K)$.*

A formula $\alpha \in K$ is called a *free formula* iff there is no $M \in MUSes(K)$ such that $\alpha \in M$. The class of free formulae of K is written *free*$(K) = K \setminus \bigcup MUSes(K)$, and its complement is named *unfree formulae* set: *unfree*$(K) = K \setminus$ *free*(K).

At the same time, we can define the *Maximal Satisfiable (Consistent) Subset* (MSS), and Hitting set as follows:

Definition 2 (MSS)**.** *Let K be a knowledge base and M a subset of K. M is a maximal satisfiable (consistent) subset (MSS) of K iff $M \nvdash \perp$ and $\forall \alpha \in K \setminus M$, $M \cup \{\alpha\} \vdash \perp$.*

Definition 3. *H is a hitting set of a set of sets Ω if $\forall S \in \Omega$, $H \cap S \neq \emptyset$. A hitting set H is irreducible (or minimal) if there is no other hitting set H' s.t $H' \subset H$. A minimum hitting set is an irreducible hitting set with the smallest cardinality.*

Many researchers have focused their effort to elaborate rational properties [19, 21, 6] that an ideal inconsistency measure I must satisfy. Among such properties we can cite the following ones:

- *Consistency:* $I(K) = 0$ iff K is consistent.
- *MinInc:* $I(M) = 1$ if $M \in MUSes(K)$.
- *Independence:* $I(K \cup \{\alpha\}) = I(K)$ if $\alpha \in$ *free*$(K \cup \{\alpha\})$.
- *Monotonicity:* if $K \subseteq K'$, then $I(K) \leq I(K')$.
- *Additivity:* $I(K_1 \cup \ldots \cup K_n) = \Sigma_{i=1}^{n} I(K_i)$ if $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \uplus \ldots \uplus MUSes(K_n)$, where $\uplus$ is a disjoint union over a family of sets.
- *Ind-Additivity:* $I(K_1 \cup \ldots \cup K_n) = \Sigma_{i=1}^{n} I(K_i)$ if $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \uplus \ldots \uplus MUSes(K_n)$, and *unfree*$(K_i) \cap$ *unfree*$(K_j) = \emptyset$ for $1 \leq i < j \leq n$.

The additivity property has attracted much more attention recently, allowing the introduction of additional conditions under which the inconsistency of the whole base can be obtained by summing up the inconsistency of its sub-bases i.e. by decomposing the whole base into sub-bases satisfying some requirements. In [19], the condition simply states that the MUSes of the knowledge base form a partition of the considered base (additivity). Such condition has been revisited in [21] by requiring a second condition over the independence of the formulae of the considered knowledge bases leading to the so called Ind-additivity. This last property allows to define the lower bound for standard inconsistency measures. Such lower bound has been characterized using the notion of closed set packing of MUSes (see Definition 4).

Definition 4 (Closed Set Packing of MUSes)**.** *Let $S = \{M_1, \ldots, M_n\}$ a subset of MUSes of K. S is a closed set packing of MUSes of K iff:*

- $\forall M_i, M_j \in S$, $M_i \cap M_j = \emptyset$
- $MUSes(M_1 \cup \ldots \cup M_n) = S$

Closed Set Packing of MUSes of K [23] is defined as a set packing of MUSes (pairwise disjoint set of MUSes) that is closed by union. The closure means that the only MUSes that can be built using the formulae involved in the union of the elements of S is S itself.

Maximum cardinality of closed set packings of MUSes of K namely $\mu(K)$ is proved to be a lower bound of measures satisfying MinInc, Monotony, and Ind-additivity i.e., $\mu(K) \leq I(K)$ for all K. Such lower bound has been used in [21] as a new measure called I_{CC}: $I_{CC}(K) = \mu(K)$.

3 MUS Structure: Motivation and Properties

Rational properties like ind-additivity aims to simplify the computation of the inconsistency value when some conditions on the structure of MUSes are gathered. However, it lacks properties to better discriminate between inconsistent knowledge bases that do not satisfy the requirements.

To motivate our study, let us consider all the possible interactions of three MUSes as depicted in Figure 1. The inconsistency values of each knowledge base K_i $(1 \leq i \leq 6)$ according to I_{MI} and I_{CC} are: $I_{MI}(K_i) = 3$ for all $i \in \{1, \ldots, 6\}$, $I_{CC}(K_i) = 1$ for all $i \in \{1, 5, 6\}$, $I_{CC}(K_i) = 2$ for all $i \in \{2, 3\}$, and $I_{CC}(K_4) = 3$.

Clearly, I_{MI} and I_{CC} fail to discriminate between these different knowledge bases. Indeed, I_{MI} considers all the knowledge bases $\{K_1, \ldots, K_6\}$ as equally inconsistent, while for I_{CC} measure, the two knowledge bases K_1 and K_5 admit the same inconsistency degree. This clearly highlights the inability of these measures to take into account dependencies between MUSes. Intuitively, formulas with disjoint MUSes are more inconsistent than those with connected MUSes. Indeed, if the set of MUSes are disjoint, then the minimum number of formulas that we need to remove to recover consistency coincides with I_{MI}. Consequently, a fine-grained measure should try to capture the degree of connectivity between MUSes. From the different configuration depicted in Figure 1, K_1 should be less inconsistent than K_3 for example. An observation on the interactions between MUSes in K_1 and K_5, suggests that K_1 is less inconsistent than K_5, as it contains two disjoint MUSes, contrary to K_5.

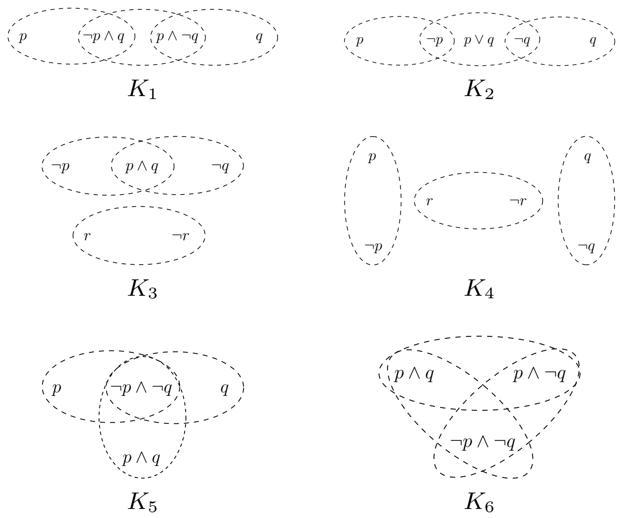

Figure 1. Interactions Between Three MUSes

Based on this observation, we propose a new property called sub-additivity in order to gain insights about the importance of the underlying structure of the MUSes.

Definition 5 (Sub-Additivity)**.** *An inconsistency measure I is called sub-additive if for any set of knowledge bases $\{K_1, \ldots, K_n\}$ s.t. $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \uplus \ldots \uplus MUSes(K_n)$ and $unfree(K_1 \cup \ldots \cup K_n) \neq unfree(K_1) \uplus \ldots \uplus unfree(K_n)$, then $I(K_1 \cup \ldots \cup K_n) < \Sigma_{i=1}^{n} I(K_i)$.*

Let us note that the condition $unfree(K_1 \cup \cdots \cup K_n) \neq unfree(K_1) \uplus \cdots \uplus unfree(K_n)$ in Definition 5, is equivalent to the following condition: there exists i and j $(1 \leq i < j \leq n)$ such that $unfree(K_i) \cap unfree(K_j) \neq \emptyset$.

The rationale behind Definition 5 is that the knowledge base with intersecting MUSes is less inconsistent than those with disjoints MUSes.

Proposition 1. *Neither I_{MI} nor I_{CC} satisfies the sub-additivity.*

Proof. 1) I_{MI}: let us consider the simple knowledge base $K = \{p, \neg p \wedge q, \neg q\}$. By considering $K_1 = \{p, \neg p \wedge q\}$ and $K_2 = \{\neg p \wedge q, \neg q\}$. We have $MUSes(K) = MUSes(K_1) \uplus MUSes(K_2)$ and $unfree(K_1) \cap unfree(K_2) \neq \emptyset$. However $I_{MI}(K) = 2 \not< I_{MI}(K_1) + I_{MI}(K_2) = 2$
2) I_{CC}: Let $K = \{p, \neg p, p \vee q, \neg q, q\}$ be a knowledge base. By considering $K_1 = \{p, \neg p, p \vee q, \neg q\}$ and $K_2 = \{\neg q, q\}$. We have $MUSes(K) = MUSes(K_1) \uplus MUSes(K_2)$ and $unfree(K_1) \cap unfree(K_2) = \{\neg q\}$ but $I_{CC}(K) = 2 \not< I_{CC}(K_1) + I_{CC}(K_2) = 2$. □

The inconsistency measures I_{MI} and I_{CC} are not the only ones violating the sub-additivity property. In fact, let us remark that hitting sets based inconsistency measure I_{hs} [36] does not satisfy sub-additivity. Indeed, let us consider $K = \{p, \neg p, p \vee q, \neg q, q\}$ where $K_1 = \{p, \neg p, p \vee q, \neg q\}$ and $K_2 = \{\neg q, q\}$. We have $I_{hs}(K) \not< I_{hs}(K_1) + I_{hs}(K_2)$ where $I_{hs}(K)$ is roughly equivalent to the cardinality of the minimum hitting set of the MUSes hypergraph. Furthermore, let us consider the MSS based inconsistency measure $I_M(K)$ defined as the maximum cardinality of the MSSs of K (see Definition 6). I_M does not satisfy sub-additivity.

Definition 6 ([14])**.** *Let K be a knowledge base. I_M is defined as:*

$$I_M(K) = |MSSs(K)| + |SelfC(K)| - 1$$

$SelfC(K)$ is the set of self contradictory formulas i.e., $\alpha \in K$ such that $\{\alpha\} \vdash \perp$.

Indeed, by considering the knowledge base $K = \{p, q, \neg p \wedge \neg q, r_1, \ldots, r_n, \neg r_1 \vee \ldots, \vee \neg r_n, s_1, \ldots, s_n, \neg s_1 \vee \ldots, \vee \neg s_n\}$ where $K_1 = \{p, \neg p \wedge \neg q, r_1, \ldots, r_n, \neg r_1 \vee \ldots \vee \neg r_n\}$ and $K_2 = \{q, \neg p \wedge \neg q, s_1, \ldots, s_n, \neg s_1 \vee \ldots, \vee \neg s_n\}$, we have $I_M(K) = 2 \times n \times n - 1$, while $I_M(K_1) = I_M(K_2) = 4 \times n - 1$. Consequently, $I_M(K) \not< I_M(K_1) + I_M(K_2)$.

Intuitively, an inconsistency measure satisfying the sub-additivity property can be defined as the one counting the number of unfree formulas.

Proposition 2. *$I_{unfree}(K) = |unfree(K)|$ satisfies the sub-additivity property.*

Proof. Let $\{K_1, K_2, \ldots, K_n\}$ be a set of knowledge bases such that $MUSes(K_1 \cup \ldots \cup K_n) = MUSes(K_1) \uplus \ldots \uplus MUSes(K_n)$ and there exists $1 \leq i \neq j \leq n$ such that $unfree(K_i) \cap unfree(K_j) \neq \emptyset$. Consequently, $I_{unfree}(K_1 \cup \cdots \cup K_n) < \sum_{i=1}^{n} I_{unfree}(K_i)$. Indeed, as $unfree(K_i) \cap unfree(K_j) \neq \emptyset$, then by summing $I_{unfree}(K_i)$ and $I_{unfree}(K_j)$, the unfree formulae shared by K_i and K_j are counted twice. □

In addition, I_{unfree} satisfies the ind-additivity, consistency, and free formula independence. However, the MinInc property is violated. This is clearly a problem, since the violation of MinInc makes I_{unfree} not enough informative about internal conflicts. Indeed, a knowledge base with several MUSes can be considered less inconsistent than a knowledge base with a single MUS.

Let us show how a measure satisfying a sub-additivity property behaves on some MUSes whose associated hypergraph is of particular structure. Let us note K^n a knowledge base with n MUSes $\{\mathsf{M}_1 \ldots, M_n\}$.We exhibit two categories of MUSes structures. The first category corresponds to MUSes hypergraph forming a star i.e. there exists a subset $S \subset K^n$ such that $\forall i (1 \leq i < j \leq n)$, $M_i \cap M_j = S$. If the MUSes hypergraph associated to K^n is a star (Figure 3.a), then the sub-additivity can be simplified using inequality (1). Indeed, if we decompose K^n to K' and K'' such that $MUSes(K^n) = MUSes(K') \uplus MUSes(K'')$, then the two hypergraphs of MUSes of K' and K'' are also stars. Consequently, there exists m such that $K' = K^m$ and $K'' = K^{n-m}$. As the MUSes hypergraphs of K' and K'' are disjoint stars, then the inconsistency degree of K^n must be less than the cumulated inconsistency degree of K' and K''.

Consequently, for knowledge bases whose MUSes hypergraph forms a star, and for any measure I considering only the number of MUSes or their interactions, satisfying sub-additivity, we derive the following inequality:

$$I(K^n) < I(K^m) + I(K^{n-m}) \tag{1}$$

The measure I satisfying the sub-additivity property obtained using inequality (1), defines a well known (strictly) sub-additive sequence of nonnegative terms $(I(K^n))$. It is known that such kind of sequences are bounded below and converges to $\frac{I(K^n)}{n} = inf\{\frac{I(K^n)}{n}, n \in \mathbb{N}\}$ [9].

Let us remark that I_{MI} has an additive behavior on a knowledge base K^n whose MUSes hypergraph forms a star. From the above discussion, measures of the form $I(K^n) = an + b$ where a and b are strictly positives integers, satisfy the sub-additivity and Monotony properties when the MUSes of K^n form a star. To satisfy $MinInc$ property, the positive integers a and b must satisfy the following constraints: $a + b = 1$, $a > 0$ and $b > 0$.

In the second category, we consider knowledge bases K^n whose MUSes $\{M_1, \ldots, M_n\}$ form a chain (Figure 3.c) i.e. $\forall i(1 \leq i < n), M_i \cap M_{i+1} \neq \emptyset$. Here again, an inconsistency measure satisfying sub-additivity property verifies inequality (1).

Similarly, measures of the form $I(K^n) = an+b$ where $a+b = 1$, $a > 0$ and $b > 0$ satisfies sub-additivity, monotony and MinInc properties. Note that, if ind-additivity is required, the value of a must be greater than $\frac{1}{2}$. Indeed, there are $\lfloor \frac{n+1}{2} \rfloor$ MUSes forming a closed set packing of MUS. Consequently, we must have $I(K_n) \geq \lfloor \frac{n+1}{2} \rfloor$. Then, $a \geq \frac{\lfloor \frac{n+1}{2} \rfloor - 1}{n-1}$, $\forall n > 2$.

We have characterized sub-additive measures for knowledge bases with MUSes hypergraph of particular structure. In the next section we provide a measure that satisfy several required properties.

4 On Sub-Additive Inconsistency Measure

In this section, we present a new measure that satisfies several standard properties, namely MinInc, Monotony, Ind-additivity, and Sub-additivity. Up to now, we argued that MUSes, MSSes, or Hitting sets based measures fail to satisfy sub-additivity property.

Let us recall that the principle behind sub-additivity is that a knowledge base with n disjoint MUSes is more inconsistent than a knowledge base with n non disjoint MUSes. Consequently, to satisfy sub-additivity property, an inconsistency measure must consider the interactions between MUSes. The main idea behind our proposed sub-additive based inconsistency measure, is to first quantify each MUS of a knowledge base with the same inconsistency value. At first, the contribution of each MUS to the whole inconsistency is equal to one. Each time two MUSes share a formula, one of them must decrease its inconsistency accordingly. In other words, there contributions to inconsistency must be different.

Before presenting our measure, we define the graph representation associated to the MUSes of a knowledge base K.

Definition 7 (MUSes Graph)**.** *Let K be a knowledge base. We define the graph representation $G^K_{mus}(V, E)$ of $MUSes(K)$ as the graph where*

- $V = MUSes(K)$ *and*
- $(M, M') \in E$ *iff* $M \cap M' \neq \emptyset$

Let K be a knowledge base and $S \subseteq MUSes(K)$, we say that S is a *MUS Cover* (MC) of K iff S is a vertex cover of $G^K_{mus}(V, E)$ i.e., $\forall(v, w) \in E$, we have $v \in S$ or $w \in S$.

Example 1. *Let us consider a MUSes hypergraph of K and its associated MUS graph depicted in Figure 2. $MC_1 = \{M_1, M_2, M_4\}$ and $MC_2 = \{M_2, M_3, M_5\}$ are two MUS cover of K.*

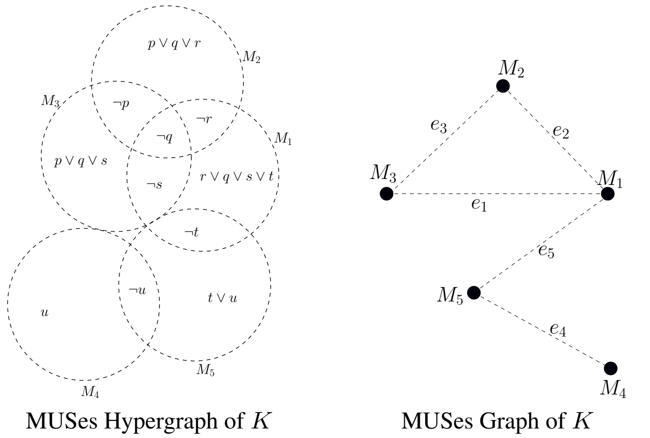

Figure 2. From MUSes Hypergraph to Graph of MUSes

It is widely known that a set of vertices is a vertex cover if and only if its complement is an independent set. So, if S is a MUS cover of K, then $MUSes(K) \setminus S$ is a pairwise disjoint set of MUSes i.e. an independent set of the graph of MUSes.

When two MUSes share formulas, their inconsistency must be set to different values in order to satisfy the sub-additivity property. To this end, let us first define an edge partition of the MUSes graph of K.

Definition 8. *Let K be a knowledge base. We define an edge partition of G^K_{mus} as:*

$$P_e = \biguplus_{M \in V} P_e(M) \text{ s.t. } P_e(M) \subseteq \{(M, M') \in E\}$$

From the definition of a partition of the edges of G^K_{mus}, each edge between two MUSes M and M' belongs either to $P_e(M)$ or to $P_e(M')$. Additionally, an edge partition P_e may contain empty sets and its size $|P_e| = |MUSes(K)|$. Such partition allows us to quantify the contribution of each MUS M as a function of the number of edges in $P_e(M)$.

Proposition 3. *Let K be a knowledge base and P_e an edge partition of G^K_{mus}. $\{M \in MUSes(K) | P_e(M) \neq \emptyset\}$ is a MUS cover of K.*

Proof. By definition, for each edge $a = (M, M')$, M or M' belongs to $\{M \in MUSes(K) | P_e(M) \neq \emptyset\}$. □

Now, we define the inconsistency measure of an edge partition as follows:

Definition 9. *Let P_e be an edge partition of G^K_{mus}. We define the inconsistency measure of P_e as:*

$$Inc^f(P_e) = \sum_{M \in V} f(|P_e(M)|)$$

where f is a strictly decreasing function over $\mathbb{N}$ such that

$$\begin{cases} f(0) = 1 \\ \lim_{n \to +\infty} f(n) = c, \quad 0 < c < 1 \end{cases}$$

As f is a strictly decreasing function over the size of the elements of the edges partition, this means that when an edge is added to

$P_e(M)$, the contribution of M to the inconsistency measure of P_e decreases. In other words, the inconsistency degree of a MUS M depends on the number of edges attributed to $P_e(M)$. It is important to observe that the maximum inconsistency value $Inc^f(P_e)$ is reached for disjoint set of MUSes. In this last case, $Inc^f(P_e) = I_{MI}$.

Different functions f can be defined. For example, one can consider $f_1(n) = 1 - c + c^{n+1}$ or $f_2(n) = 1 - c + \frac{c}{n+1}$ s.t. $0 < c < 1$.

Example 2. *Let us consider again the example depicted in Figure 2. $P_e(M_1) = \{e_1\}$, $P_e(M_2) = \{e_2\}$, $P_e(M_3) = \{e_3\}$, $P_e(M_4) = \{e_4\}$, and $P_e(M_5) = \{e_5\}$ is an edge partition. Using f_1, we have $Inc^f(P_e) = 5(1 - c + c^2)$*

Now, we are ready to define our inconsistency measure of given knowledge base.

Definition 10. *Let K be a knowledge base. The inconsistency of K is defined as:*

$$Inc(K) = \max_{P_e} \; Inc^f(P_e)$$

Let us provide the intuition behind the application of the maximum function over edge partitions. Assume we have n pairwise disjoint MUSes namely $\{M_1, \dots, M_n\}$ and an additional MUS M_{n+1} admitting a non empty intersection with the n first MUSes. Among the possible edge partitions we have P_e where $\forall i(1 \leq i \leq n)$ $P_e(M_i) = \{(M_i, M_{n+1})\}$ and $P_e(M_{n+1}) = \emptyset$. Another edge partition P'_e can be defined as $\forall i(1 \leq i \leq n)$, $P'_e(M_i) = \emptyset$ and $P'_e(M_{n+1}) = \{(M_1, M_{n+1}), \dots, (M_n, M_{n+1})\}$. We have $Inc^f(P_e) = n + 1 - f(1)n$ and $Inc^f(P'_e) = n + 1 - f(n)$. As monotony is required, we should have $Inc(K) \geq n$. $Inc^f(P_e)$ fails to satisfy such requirement when n is large enough. In contrast, with P'_e, the monotony property is satisfied. As it will be proved later, choosing the maximum inconsistency value among all the edge partitions is a key point towards the satisfaction of all the required properties.

Example 3. *Let us consider again the example of Figure 2. Using f_1, there are two edge partitions maximizing the inconsistency. The first one is P_e defined as follows: $P_e(M_1) = \{e_1, e_2, e_5\}$, $P_e(M_2) = \{e_3\}$, $P_e(M_3) = \emptyset$, $P_e(M_4) = \{e_4\}$, and $P_e(M_5) = \emptyset$. The second edge partition P'_e can be obtained from P_e by permuting M_2 and M_3. The maximum value is obtained either with P_e or P'_e:*

$$Inc(K) = 5 - 3c + 2c^2 + c^3$$

Let us now take another example described in Section 3 (see Figure 1). Applying our new measure, we obtain:

$$Inc(K_1) = 3 - c + c^3 \qquad Inc(K_2) = 3 - c + c^3$$

$$Inc(K_3) = 3 - c + c^2 \qquad Inc(K_4) = 3$$

$$Inc(K_5) = 3 - 2c + c^2 + c^3 \qquad Inc(K_6) = 3 - 2c + c^2 + c^3$$

Our inconsistency measure allows to reorder the set of knowledge bases $\{K_1, \dots, K_6\}$ in the following way:

$$Inc(K_5) = Inc(K_6) < Inc(K_1) = Inc(K_2) < Inc(K_3) < Inc(K_4)$$

As we can see, K_4 is the most inconsistent one, while K_5 and K_6 are the least inconsistent ones. Furthermore $Inc(K_1) < Inc(K_3)$ and $Inc(K_2) < Inc(K_3)$ are required to satisfy the sub-additivity property. Our measure do not make distinction between K_5 and K_6 or K_1 and K_2. Indeed, the MUSes graph of K_5 and K_6 are very similar, as well as for K_1 and K_2.

We now show that Inc satisfy all required properties.

Proposition 4. *Inc satisfies Consistency, Free Formula Independence, MinInc, Monotony, Ind-Additivity and Sub-Additivity properties.*

Proof.

Consistency: If the knowledge base is consistent, the only partition P_e is the empty set. According to Definition 9, $Inc^f(P_e) = 0$. Consequently, $Inc(K) = 0$.

Free Formula Independence: Our measure consider only the MUSes, so the free formula independence is satisfied i.e., Inc(K) = Inc(unfree(K)).

MinInc: For a knowledge base K with a single MUS M, we have a single edge partition P_e s.t. $P_e(M) = \emptyset$. Consequently $Inc(K) = 1$.

Monotony: Let K be a knowledge base and α a formula. There is two cases:

- $\alpha \in free(K)$: $Inc(K) = Inc(K \cup \{\alpha\})$
- $\alpha \in unfree(K)$: Let $\{M_1, \dots, M_n\} = MUSes(K)$, $\{M'_1, \dots, M'_m\} = MUSes(K \cup \{\alpha\}) \setminus MUSes(K)$ and P_e an edge partition of K. P_e can be extended to an edge partition P'_e of $K \cup \{\alpha\}$. P'_e is such that if $a = (M_i, M'_j) \in E$ or $a = (M'_i, M'_j) \in E$, then $a \in P'_e(M'_j)$. Consequently, $Inc^f(P'_e) = Inc^f(P_e) + \sum_{i=1}^{m} f(M'_i)$. As $f(M'_i) > 0$ for all $0 < i < m$, then $Inc^f(P'_e) > Inc^f(P_e)$. Consequently, $Inc(K \cup \{\alpha\}) > Inc(K)$.

Ind-Additivity: If $\{K_1, \dots, K_n\}$ satisfies the conditions of the application of ind-Additivity then, each P_e an edge partition of $K_1 \cup \dots \cup K_n$ can be decomposed into disjoint edge partitions of $K_1, K_2, \dots,$ and K_n. Consequently, $Inc(K_1 \cup \dots \cup K_n) = \sum_{i=1}^{n} Inc(K_i)$

Sub-Additivity: Let $K_1, \dots, K_n$ such that $MUSes(K_1 \cup \dots \cup K_n) = \biguplus_{i=1}^{n} MUSes(K_i)$ and there exists i, j such that $K_i \cap K_j \neq \emptyset$. Performing $\sum_{i=1}^{n} Inc(K_i)$ is equivalent to remove in the MUSes graph G_{mus} of $(K_1 \cup \dots \cup K_n)$ the edges S linking the MUSes of K_i to MUSes of K_j. Let us note G'_{mus} such obtained graph. Let P_e be an edge partition of G_{mus}. Let us consider P'_e such that if $P_e(M) \in P_e$ then $P_e(M) \setminus S \in P'_e$. P'_e is an edge partition of G'_{mus}. As $S \neq \emptyset$, we have $Inc^f(P_e) < Inc^f(P'_e)$. Finally, P'_e is an edges partition of G'_{mus}, $Inc(K_1 \cup \dots \cup K_n) < \sum_{i=1}^{n} Inc(K_i)$. □

Finally, in Proposition 5, we show that our inconsistency measure is bounded by the cardinality of the maximum independent set of the graph of MUSes.

Proposition 5. *Let K be a knowledge base. We have*

$$Inc(K) \geq max_{S \in IS(G^K_{mus})}(|S|)$$

where $IS(G^K_{mus})$ is the set of independent sets of G^K_{mus}.

Proof. Let S be an independent set of G^K_{mus}. Then, there is no M, M' in S such that $M \cap M' \neq \emptyset$. Then it is possible to build an edge partition P_e such that $P_e(M) = \emptyset$ for all $M \in S$. It is sufficient to attribute an edge (M, M') to $P_e(M')$ if $M \in S$ and $M' \in MUSes(K) \setminus S$. As $f(M) = 1$ when $P_e(M) = \emptyset$, then *Inc*$(K) \geq max_{S \in IS(G^K_{mus})}(|S|)$. □

5 Discussion

Let us analyze the behavior of *Inc* inconsistency measure through its MUSes dependencies. We focus our study on three hypergraph classes: stars and chains (Figure 3.a and 3.c), and near disjoint (Figure 3.b). As f, we use the f_1 function defined in Section 4.

Stars: Let K a knowledge base with a set of MUSes $\{M_1, \ldots, M_n\}$. We suppose that the MUSes hypergraph of K is a star. The MUSes graph G^K_{mus} is a clique. In this case, one edge partition P_e maximizing *Inc*(K) is: $P_e(M_1) = \emptyset$ and $P_e(M_i) = \{(M_i, M_1), \ldots, (M_i, M_{i-1})\}$. By simplifying *Inc*$(K)$, we obtain the following result:

$$Inc(K) = Inc^{f_1}(P_e) = (1-c)n + c + \frac{c^2}{1-c}(1 - c^{n-1})$$

Let us note that asymptotically, for large number of MUSes, the average contribution of each MUS tends to $(1-c)$ since $\lim\limits_{n \to +\infty} \frac{Inc(K)}{n} = 1 - c$.

Chains: Suppose now that G^K_{mus} is a chain and contains an odd number of nodes. The partition P_e maximizing *Inc*$^{f_1}(P_e)$ is: $P_e(M_i) = \emptyset$, if $i = 2k + 1$ and $P_e(M_i) = \{(M_{i-1}, M_i), (M_i, M_{i+1})\}$ if $i = 2k$. Then, the inconsistency value can be expressed as:

$$Inc(K) = Inc^{f_1}(P_e) = 1 + (1 - \frac{c}{2} + \frac{c^3}{2})(n-1)$$

Asymptotically, we obtain $\lim\limits_{n \to +\infty} \frac{Inc(K)}{n} = 1 - \frac{c}{2} + \frac{c^3}{2}$, the average contribution of each MUS to the inconsistency when n is large enough. It is worth noticing that, for chains MUSes hypergraph, the average contribution of each MUS is higher than for stars MUSes hypergraph ($1 - \frac{c}{2} + \frac{c^3}{2} > 1 - c$). This is rational since chains are less connected than stars (MUS graph is a clique). Similar reasoning can be applied for G^K_{mus} with an even number of nodes.

Near disjoint: Let us consider the MUSes $\{M_1, \ldots, M_{n+1}\}$ of a knowledge base K such that $\{M_1, \ldots, M_n\}$ are pairwise disjoint and M_{n+1} has a non empty intersection with a subset $S \subseteq \{M_1, \ldots, M_n\}$ such that $|S| = k$. This near disjoint MUSes hypergraph is depicted in Figure 3).b. We obtain:

$$Inc(K) = n + 1 - c + c^{k+1}$$

For a fixed value of n, *Inc* suggests that the inconsistency value grows inversely to that of k. In other words, the inconsistency grows as the connections between MUSes decreases.

6 Computing Inconsistency Value

In this section, we are interested in computing *Inc*(K). We provide a formulation as an optimization problem allowing the computation of *Inc*(K), when the function f is fixed (see Definition 10 and 9). Let us recall that our approach is based on a partition of the set of edges over the set of vertices (MUSes) such that an edge $e = (M, M')$ is attributed either to M or to M'. The contribution of each MUS depends on the number of edges attributed to it. Formally, to seek for an optimal solution, we associate to each edge $e = (M, M')$ of G^K_{mus}, a new variable x_e. If e is attributed to M then x_e is true, otherwise it is false and it is associated with M'. This is equivalent to the attribution of x_e to M and $\neg x_e$ to M'. Then the global formulation can be obtained by considering the function used in Definition 9.

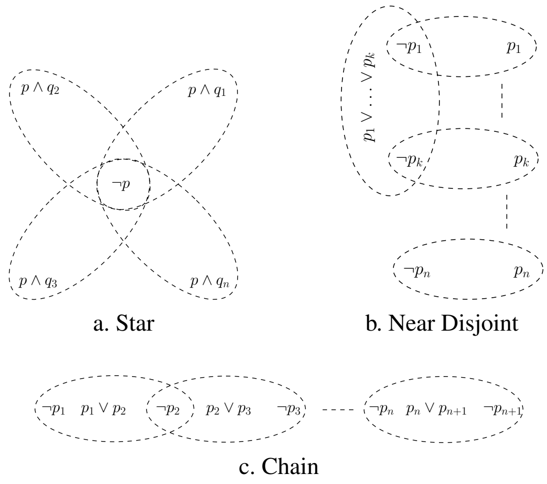

Figure 3. MUSes Hypergraph Classes

To illustrate the formulation of Definition 10 as an optimization problem, we consider the example of Figure 2. Let x_{e_i} ($1 \leq i \leq 5$) be the variable associated to edge e_i as depicted in Figure 2 (right hand side).

Using the function f_1, computing *Inc*(K), is equivalent to the maximization of the objective function depicted in Equation (2) obtained in the following way. We recall that $f_1(n) = 1 - c + c^{n+1}$. Let $M_1 \prec M_2 \prec \ldots \prec M_5$ a total ordering on the set of MUSes and $\Gamma_e(M)$ the set of edges connected to M in the MUSes graph. The edge partition can be expressed as $P_e(M_i) = \{e \in \Gamma_e(M_i) | e \notin P_e(M_j), j < i\}$. In our example, the size of the elements of the edges partition can then be expressed as follows: $|P_e(M_1)| = x_{e_1} + x_{e_2} + x_{e_5}$, $|P_e(M_2)| = \neg x_{e_2} + x_{e_3}$, $|P_e(M_3)| = \neg x_{e_1} + \neg x_{e_3}$, $|P_e(M_4)| = x_{e_4}$ and $|P_e(M_5)| = \neg x_{e_4} + \neg x_{e_5}$. Then the objective function (Equation 2) can be derived from $Inc^f(P_e) = \sum_{i=1}^{5} f_1(|P_e(M_i)|)$.

Similarly, for the function f_2, we obtain the objective function described in Equation (3). Let us remark that the expression of Definition 10 as an optimization problem leads to a nonlinear objective function as illustrated by both Equations (2) and (3). Solving nonlinear optimization problems is generally more difficult than for linear problems. One can prove that Equation 2 can be linearized. A classical approach to find such optimum is to use a branch and bound like procedure.

As using an exact optimization method is computationally more costly, we propose in the sequel, an approximation of the *Inc*(K) value. If the set of MUSes can be obtained in reasonable amount of time, one can use minimum vertex cover problem as an approximation of *Inc*(K). Indeed, to maximize $Inc(K)$, one have to minimize the set of MUSes M with $P_e(M) = \emptyset$ which is equivalent to the problem of finding the minimum vertex cover of the MUSes graph. Computing the minimum vertex cover is known to be an NP-Hard

2.
$$Inc^{f_1}(P_e) = 5(1-c) + c^{1+x_{e_1}+x_{e_2}+x_{e_5}} + c^{1+\neg x_{e_2}+x_{e_3}} + c^{1+\neg x_{e_1}+\neg x_{e_3}} + c^{1+x_{e_4}} + c^{1+\neg x_{e_4}+\neg x_{e_5}} \quad (2)$$

$$Inc^{f_2}(P_e) = 5(1-c) + \frac{1}{1+x_{e_1}+x_{e_2}+x_{e_5}} + \frac{1}{1+\neg x_{e_2}+x_{e_3}} + \frac{1}{1+\neg x_{e_1}+\neg x_{e_3}} + \frac{1}{1+x_{e_4}} + \frac{1}{1+\neg x_{e_4}+\neg x_{e_5}} \quad (3)$$

problem. It can be formulated easily as a linear program.

Problem: Minimum Vertex Cover

$minimize \sum_{v \in V} x_v$ (minimize the total cost)

subject to

$x_u + x_v \geq 1 \; \forall \{u, v\} \in E$ (cover every edge)

$x_v \in \{0, 1\} \; \forall v \in V$ ($x_v = 1 \Leftrightarrow$ v in the vertex cover)

Another alternative consists in approximating the minimum vertex cover using a greedy approach as shown in Algorithm 1. This algorithm is linear and allows to compute a reasonable approximation of the value of *Inc*(K). For instance, using the greedy algorithm, the value obtained for the knowledge base of Figure 2 corresponds exactly to the optimal inconsistency value *Inc*(K).

Algorithm 1 Inc(K): Greedy Approximation Approach

Require: A graph $G^k_{mus} = (V, E)$
Ensure: An edge partition P_e

```
P_e ← ∅
val ← 0
for M ∈ V do
    E_d(M) ← {e | e = (M, M') ∈ E}
end for
repeat
    node ← max_{M∈V} |E_d(M)|
    E ← E \ P_e(M)
    V ← V \ {node}
    val ← val + f(|P_e(M)|)
    P_e(node) ← E_d(node)
    for M' ∈ V do
        P_e(M') ← P_e(M') \ P_e(node)
    end for
until (V = ∅)
return val
```

Algorithm 1 describes a greedy approach that aims to find an approximation of *Inc*(K) based on vertex cover. First, the edges are distributed over vertices (MUSes). If $e = (M, M') \in E$, then e is associated temporarily to M and M'. At each iteration, the most connected vertex $node$ is chosen and P_e is updated accordingly. Then, the edges of this node are removed. The process is iterated until the set of vertices V becomes empty.

7 Related Work

In this section, we provide a brief overview of some works related to inconsistency measures. Several inconsistency measures have been proposed over the years. An interesting work has been performed recently in [37] to compare a large set of inconsistency measures in terms of their ability to discriminate between knowledge bases. However, it is common to partition the set of approaches into three separate classes. The first one includes those based on either minimal inconsistent subsets [19, 28, 29, 21, 1, 24], or maximal consistent subsets [13, 8]. The second one [10, 16, 17, 30, 18, 12, 27, 38, 26, 15], focus on the semantics of the language, often based on some multi-valued semantics [32]. For example, in [39], the authors take the ratio of the propositions appearing in a minimal inconsistent subset wrt. the total number of propositions as the inconsistency value. The third one is based on probabilistic models [25, 8].

Among original approaches, one can cite also the one of [19]. It exploits the Shapley value, originally introduced in cooperative game, to analyze and quantify the amount of inconsistency that can be imputed to each formula in a given knowledge base. Usually, inconsistency measures can be partitioned according to their dependence on syntax or semantics. Semantic based measures aim to compute the proportion of the language that is affected by the inconsistency. The inconsistency measures belonging to this class are often based on some paraconsistent semantics and, thus, syntax independent, because we can still find paraconsistent models for inconsistent KBs. Whilst, syntax based approaches are concerned with the minimal number of formulas that cause inconsistencies. An overview of inconsistency measures for classical logics can be found in [13].

There is also related work on inconsistency measures in the context of quantitative logics. In particular, several works have extended existing inconsistency measures for classical frameworks to the probabilistic setting, while investigating their properties. One can quote for example the family of inconsistency metrics, proposed by [25, 8, 31, 35], based on the quantification of the minimal adjustments in the degrees of certainty (i.e., probabilities) of the statements necessary to make the knowledge base consistent. In [34], another inconsistency measure for probabilistic conditional logic is proposed. It is based on generalized divergence which is a specific distance for probability functions.

8 Conclusion and Future Work

In this paper we developed a novel approach to measure inconsistency in a knowledge base. A new property called sub-additivity is introduced providing a way to finely reorder knowledge bases. We showed that the classical approaches based on MUSes and their variants fail to satisfy the sub-additivity property. Then, we propose a new measure that exploits connections between MUSes and satisfies several properties including sub-additivity.

Our results are clearly of great interest. First, sub-additivity push further the issue of comparing the inconsistency of different knowledge bases. Secondly, MUS dependencies have been proven to be a key point for the design of more rational inconsistency measures. Several challenges need to be tackled in the future. Finding a better approximation of the inconsistency measure without enumerating all the MUSes is an important research issue. Intersections between MUSes can be more finely analyzed to improve the proposed measure. Finally, as a short term issue, we plan to analyze our measures in the light of the rational properties proposed in [6].

REFERENCES

[1] Meriem Ammoura, Badran Raddaoui, Yakoub Salhi, and Brahim Oukacha, 'On measuring inconsistency using maximal consistent sets', in *ECSQARU*, pp. 267–276, (2015).

[2] Samantha Bail, Matthew Horridge, Bijan Parsia, and Ulrike Sattler, 'The justificatory structure of the NCBO bioportal ontologies', in *Proceedings of ISWC'2011*, pp. 67–82, (2011).

[3] Ana Belén Barragans-Martinez, José Pazos Arias, and Ana Fernandez Vilas, 'On measuring levels of inconsistency in multi-perspective requirements specifications', in *PRISE*, pp. 21–30, (2004).

[4] Salem Benferhat, Didier Dubois, and Henri Prade, 'A local approach to reasoning under inconsistency in stratified knowledge bases', in *ECSQARU*, pp. 36–43, (1995).

[5] Salem Benferhat and Laurent Garcia, 'Handling locally stratified inconsistent knowledge bases', *Studia Logica*, **70**(1), 77–104, (2002).

[6] Philippe Besnard, 'Revisiting postulates for inconsistency measures', in *JELIA*, pp. 383–396, (2014).

[7] Sylvie Coste-Marquis and Pierre Marquis, 'Recovering consistency by forgetting inconsistency', in *JELIA*, pp. 113–125, (2008).

[8] Dragan Doder, Miodrag Raskovic, Zoran Markovic, and Zoran Ognjanovic, 'Measures of inconsistency and defaults', *Int. J. Approx. Reasoning*, **51**(7), 832–845, (2010).

[9] M. Fekete, 'Über die verteilung der wurzeln bei gewissen algebraischen gleichungen mit ganzzahligen koeffizienten', *Mathematische Zeitschrift*, **17**(1), 228–249, (1923).

[10] John Grant, 'Classifications for inconsistent theories', *Notre Dame Journal of Formal Logic*, **19**(3), 435–444, (1978).

[11] John Grant and Anthony Hunter, 'Measuring inconsistency in knowledgebases', *J. Intell. Inf. Syst.*, **27**(2), 159–184, (2006).

[12] John Grant and Anthony Hunter, 'Analysing inconsistent first-order knowledgebases', *Artif. Intell.*, **172**(8-9), 1064–1093, (2008).

[13] John Grant and Anthony Hunter, 'Measuring consistency gain and information loss in stepwise inconsistency resolution', in *ECSQARU*, pp. 362–373, (2011).

[14] John Grant and Anthony Hunter, 'Measuring the good and the bad in inconsistent information', in *IJCAI*, pp. 2632–2637, (2011).

[15] John Grant and Anthony Hunter, 'Distance-based measures of inconsistency', in *ECSQARU*, pp. 230–241, (2013).

[16] Anthony Hunter, 'Measuring inconsistency in knowledge via quasi-classical models', in *AAAI*, pp. 68–73, (2002).

[17] Anthony Hunter, 'Evaluating significance of inconsistencies', in *IJCAI*, pp. 468–478, (2003).

[18] Anthony Hunter, 'How to act on inconsistent news: Ignore, resolve, or reject', *Data Knowl. Eng.*, **57**(3), 221–239, (2006).

[19] Anthony Hunter and Sébastien Konieczny, 'On the measure of conflicts: Shapley inconsistency values', *Artif. Intell.*, **174**(14), 1007–1026, (2010).

[20] Anthony Hunter, Simon Parsons, and Michael Wooldridge, 'Measuring inconsistency in multi-agent systems', *Unstliche Intelligenz*, **28**(3), 169–178, (2014).

[21] Said Jabbour, Yue Ma, and Badran Raddaoui, 'Inconsistency measurement thanks to MUS-decomposition', in *AAMAS*, pp. 877–884, (2014).

[22] Saïd Jabbour, Yue Ma, Badran Raddaoui, and Lakhdar Saïs, 'Prime implicates based inconsistency characterization', in *ECAI*, pp. 1037–1038, (2014).

[23] Saïd Jabbour, Yue Ma, Badran Raddaoui, Lakhdar Sais, and Yakoub Salhi, 'On structure-based inconsistency measures and their computations via closed set packing', in *AAMAS*, pp. 1749–1750, (2015).

[24] Saïd Jabbour, Badran Raddaoui, and Lakhdar Sais, 'Inconsistency-based ranking of knowledge bases', in *ICAART*, pp. 414–419, (2015).

[25] Kevin Knight, 'Measuring inconsistency', *J. Philosophical Logic*, **31**(1), 77–98, (2002).

[26] Yue Ma, Guilin Qi, and Pascal Hitzler, 'Computing inconsistency measure based on paraconsistent semantics', *J. Log. Comput.*, **21**(6), 1257–1281, (2011).

[27] Yue Ma, Guilin Qi, Guohui Xiao, Pascal Hitzler, and Zuoquan Lin, 'Computational complexity and anytime algorithm for inconsistency measurement', *Int. J. Software and Informatics*, **4**(1), 3–21, (2010).

[28] Kedian Mu, Weiru Liu, and Zhi Jin, 'A general framework for measuring inconsistency through minimal inconsistent sets', *Knowl. Inf. Syst.*, **27**(1), 85–114, (2011).

[29] Kedian Mu, Weiru Liu, and Zhi Jin, 'Measuring the blame of each formula for inconsistent prioritized knowledge bases', *J. Log. Comput.*, **22**(3), 481–516, (2012).

[30] Carlos A. Oller, 'Measuring coherence using lp-models', *J. Applied Logic*, **2**(4), 451–455, (2004).

[31] David Picado-Muiño, 'Measuring and repairing inconsistency in probabilistic knowledge bases', *Int. J. Approx. Reasoning*, **52**(6), 828–840, (2011).

[32] Graham Priest, 'The logic of paradox', *J. Philosophical Logic*, **8**(1), 219–241, (1979).

[33] Guilin Qi, Weiru Liu, and David A. Bell, 'Measuring conflict and agreement between two prioritized belief bases', in *IJCAI*, pp. 552–557, (2005).

[34] W. Rodder and L. Xu, 'Elimination of inconsistent knowledge in the probabilistic expert system-shell SPIRIT (in German)', in *Operations Research Proceedings - Springer-Verlag*, pp. 260–265, (2001).

[35] Matthias Thimm, 'Inconsistency measures for probabilistic logics', *Artif. Intell.*, **197**, 1–24, (2013).

[36] Matthias Thimm, 'Towards large-scale inconsistency measurement', in *KI 2014*, pp. 195–206, (2014).

[37] Matthias Thimm, 'On the expressivity of inconsistency measures', *Artif. Intell.*, **234**, 120–151, (2016).

[38] Guohui Xiao, Zuoquan Lin, Yue Ma, and Guilin Qi, 'Computing inconsistency measurements under multi-valued semantics by partial Max-SAT solvers', in *KR*, (2010).

[39] Guohui Xiao and Yue Ma, 'Inconsistency measurement based on variables in minimal unsatisfiable subsets', in *ECAI*, pp. 864–869, (2012).