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On Redundancy in Simple Temporal Networks[1]

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Abstract. The Simple Temporal Problem (STP) has been widely used in various applications to schedule tasks. For dynamical systems, scheduling needs to be efficient and flexible to handle uncertainty and perturbation. To this end, modern approaches usually encode the temporal information as an STP instance. This representation contains redundant information, which can not only take a significant amount of storage space, but also make scheduling inefficient due to the non-concise representation. In this paper, we investigate the problem of simplifying an STP instance by removing redundant information. We show that such a simplification can result in a unique minimal representation without loss of temporal information, and present an efficient algorithm to achieve this task. Evaluation on a large benchmark dataset of STP exhibits a significant reduction in redundant information for the involved instances.

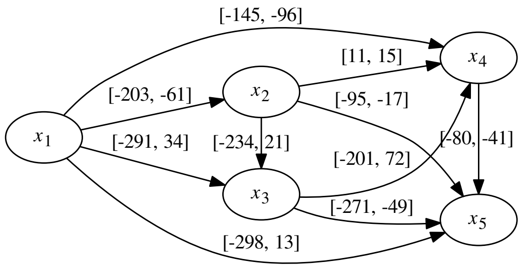
1 INTRODUCTION

The ability to reason about temporal information is necessary for intelligent agents that plan their actions to achieve their goals optimally. As such, temporal reasoning has been an active research area in Artificial Intelligence [6, 2, 16, 3, 17].

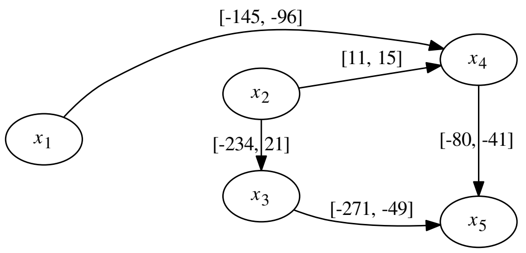
Among the different temporal representation frameworks, some of the most well-known ones are based on relations between time points or time intervals. Prominent examples of such representation frameworks include the Simple Temporal Problem [5], Allen's Interval Algebra [1], and Point Algebra [20]. In this paper, we will focus on the Simple Temporal Problem and be concerned with redundant information in instances of this problem.

The *Simple Temporal Problem* (STP) encodes the quantitative difference between two variables representing time points. An STP constraint $(x\ [a, b]\ y)$ associates an interval $[a, b]$ with variables x and y to represent the lower and upper bounds of the difference, i.e. $a \leq y - x \leq b$. An STP instance is called a Simple Temporal Network (STN), and consists of a set of variables and a set of STP constraints involving those variables. This can be represented as a labelled directed graph (see Figure 1). A solution of an STN is an assignment of time points to the variables such that all of the corresponding STP constraints are satisfied by the assignment.

There are often redundant constraints in STNs. Figure 1a shows an STN extracted from the job-shop scheduling problem benchmark dataset used in [14]. (We note that in all our examples we have modified the original STNs by reducing the bounds in their constraints to make the examples easier to follow; however, all qualitative properties of the STNs have been retained). We observe that



(a) An STN extracted from a job-shop problem instance.



(b) An STN obtained by removing all redundant constraints in the STN in Figure 1a. The two STNs are equivalent by Theorem 6.

Figure 1. Two equivalent STNs.

the STN is densely structured and contains redundant constraints that can be removed without affecting the set of solutions. For example, the constraint $(x_2\ [-95, -17]\ x_5)$ is redundant and can be removed, because combining the constraint between x_2 and x_4 and the constraint between x_4 and x_5 implies a tighter constraint, viz., $(x_2\ [-69, -26]\ x_5)$. Redundant constraints are not limited to the aforementioned trivial case; although the constraint between x_3 and x_4 cannot be directly inferred from any other two constraints, it can be inferred by combining more constraints (cf. Example 6 in Section 3). Identifying such redundant constraints efficiently is one of the main goals of this paper.

After identifying all redundant constraints in an STN, the question arises whether we can remove all of them, while maintaining the solution set unchanged. The answer is not straightforward as the redundant constraints can depend on each other, in the sense that removing

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one redundant constraint can make another constraint non-redundant. In this paper we characterize STNs that retain the solution sets after removing all redundant constraints. As an example, the STN in Figure 1a and the STN in Figure 1b that results from removing all redundant constraints of the former STN share the same solution set.

In the literature, the problem of identifying and removing redundant information has been discussed in the context of logic formulas [8, 10] and of qualitative constraint networks (QCNs) [9, 18]. For QCNs, it was shown in [9] that an all-different network defined over a distributive subclass of relations of RCC8 [15] or of Point Algebra [20] has a unique subset of non-redundant constraints that characterises a network that is equivalent to the original one. Efficient algorithms have been developed to identify such subsets [9, 18]. Since QCNs are defined over a finite set of qualitative relations, whereas STNs are defined on quantitative relations, the techniques developed in [9, 18] are not immediately applicable to STNs.

For STP constraints, [11] and [4] identify another notion of redundancy, so-called *dominance*, for inferring the range of a variable. By identifying and removing such redundancy, a particular property of an STN, called the *dispatchability* [11], which is helpful for generating solutions of an STN online, may be retained. By constrast, in this paper we will show that, by removing redundant constraints, the structure of an STN is simplified, while the solution set remains the same.

The remainder of the paper is organised as follows. In Section 2, we recall some basic concepts and formally define redundant constraints. Then we prove some properties of redundant constraints in STNs in Section 3, and use those properties to devise an algorithm for identifying and removing redundant constraints in Section 4. Section 5 evaluates our approach by using benchmark datasets. Finally, Section 6 concludes the paper.

2 PRELIMINARIES

The *simple temporal problem* (STP) is a constraint satisfaction problem where each constraint is a set of linear inequalities of the form

$$a \leq y - x \leq b, \tag{1}$$

where a, b are constants and x, y are variables defined on a continuous domain representing time points. The constraint in (1) is abbreviated as $(x\ [a,b]\ y)$. As (1) is equivalent to $-b \leq x - y \leq -a$, we also obtain $(y\ [-b,-a]\ x)$.

Algebraic operations on STP constraints are defined as follows. The *intersection* of two STP constraints defined on variables x, y yields a new constraint over x, y that represents the conjunction of the constraints. It is defined as $(x\ [a_1,b_1]\ y) \cap (x\ [a_2,b_2]\ y) := (x\ [c,d]\ y)$, where $[c,d] = [a_1,b_1] \cap [a_2,b_2]$.

The *composition* of an STP constraint over variables x, z and another STP constraint z, y yields a new STP constraint over x, y that is inferred from the other two constraints and is defined as $(x\ [a_1,b_1]\ z) \otimes (z\ [a_2,b_2]\ y) := (x\ [c,d]\ y)$, where $c = a_1 + a_2$ and $d = b_1 + b_2$.

Note that for STP constraints, the composition and intersection are associative and, as noted in [5], composition distributes over non-empty intersection, i.e., $(x\ [a,b]\ y) \otimes \big((y\ [c,d]\ z) \cap (y\ [e,f]\ z)\big) = \big((x\ [a,b]\ y) \otimes (y\ [c,d]\ z)\big) \cap \big((x\ [a,b]\ y) \otimes (y\ [e,f]\ z)\big)$.

Definition 1. An instance of STP is called a *simple temporal network* (STN) and is a tuple (V, Γ), where $V = \{x_1, \ldots, x_n\}$ is a set of variables and Γ is a set of STP constraints defined on V.

We assume that all variables in V appear in Γ, thus, we will simply use Γ to refer to an STN and not explicitly mention V. Furthermore, we have that there is only one constraint between x and y.

An STN naturally induces a graph in the following sense.

Definition 2. The *constraint graph* $G = (V, E)$ of an STN Γ is an undirected graph, where the set V of vertices consists of the variables in Γ and the set E of edges consists of constrained unordered pairs of variables in Γ, i.e.,

$$E = \{\{x, y\} \mid x, y \in V, x \neq y, (x\ [a,b]\ y) \in \Gamma\}.$$

For simplicity, we write xy for an edge in place of $\{x, y\}$.

Let $G = (V, E)$ be the constraint graph of an STN Γ. For variables x, y with $xy \in E$, we write $\Gamma(x, y)$ to refer to the constraint between x and y in Γ. Thus, if $\Gamma(x, y) = (x\ [a,b]\ y)$, then $\Gamma(y, x) = (y\ [-b,-a]\ x)$. Note that in Figure 1a we used a labelled *directed* graph to illustrate an STN, where for any edge $xy \in E$ with $\Gamma(x, y) = (x\ [a,b]\ y)$ there is exactly one directed edge (x, y) in the illustration with the corresponding interval $[a, b]$.

Definition 3. A *solution* of an STN Γ is an assignment of time points to the variables in Γ such that all constraints in Γ are satisfied; in this case we say that Γ is *consistent*. Given two STNs Γ and Γ' defined on the same set of variables we write $\Gamma' \models \Gamma$, if every solution of Γ' is a solution of Γ. If $\Gamma' \models \Gamma$ and $\Gamma \models \Gamma'$ then we say that they are equivalent and write $\Gamma' \equiv \Gamma$. We also write $\Gamma \models (x\ [a,b]\ y)$ if every solution of Γ satisfies $(x\ [a,b]\ y)$.

We observe that $\{(x\ [a,b]\ y)\} \models (x\ [c,d]\ y)$ if and only if $[a,b] \subseteq [c,d]$. Therefore, if $\{(x\ [a,b]\ y)\} \models (x\ [c,d]\ y)$, we will say that $(x\ [a,b]\ y)$ *refines* $(x\ [c,d]\ y)$, written as $(x\ [a,b]\ y) \subseteq (x\ [c,d]\ y)$. We call an STN Γ' a *refinement* of Γ, if Γ' and Γ are defined on the same set of variables and if for every constraint $(x\ [a,b]\ y) \in \Gamma$ there exists a constraint $(x\ [a',b']\ y) \in \Gamma'$ that refines $(x\ [a,b]\ y)$.

Definition 4 (Minimality)**.** Let Γ be a consistent STN and let x and y be variables in Γ. Then an STP constraint $(x\ [a,b]\ y)$ is said to be *minimal* in Γ, if $\Gamma \models (x\ [a,b]\ y)$ and $(x\ [a,b]\ y)$ is the smallest constraint with respect to $\subseteq$.

A refinement Γ^{m} of Γ is said to be *the minimal network of* Γ, if for all $x, y \in V$ with $x \neq y$ there is a constraint between x and y in Γ^{m} that is minimal in Γ.

Note that Γ^{m} is uniquely defined, and since $\Gamma^{\mathrm{m}} \models \Gamma$ and $\Gamma \models \Gamma^{\mathrm{m}}$, we have $\Gamma^{\mathrm{m}} \equiv \Gamma$. Figure 2a shows the minimal network of the STN Γ in Figure 1a, and it is also the minimal network of the STN in Figure 1b. We note that if two STNs Γ_1 and Γ_2 are equivalent, then they have the same minimal network: Suppose $\Gamma_1^{\mathrm{m}}(x, y) \neq \Gamma_2^{\mathrm{m}}(x, y)$ are minimal in Γ_1 and Γ_2, respectively, then $\Gamma_1^{\mathrm{m}}(x, y) \cap \Gamma_2^{\mathrm{m}}(x, y)$ is minimal in both Γ_1 and Γ_2 and is properly contained in either $\Gamma_1^{\mathrm{m}}(x, y)$ or $\Gamma_2^{\mathrm{m}}(x, y)$, contradicting our assumption.

Definition 5. Let $G = (V, E)$ be the constraint graph of an STN Γ. A sequence $P := (x_{i_0}x_{i_1}, x_{i_1}x_{i_2}, \ldots, x_{i_{k-1}}x_{i_k})$ of edges in E is called a *path* from x_{i_0} to x_{i_k}. The length of P is the number of edges in P and is denoted by $|P|$. A path P is called a *cycle*, if $x_{i_k} = x_{i_0}$. By $\Gamma(P)$ we denote the set $\{\Gamma(x_{i_0}, x_{i_1}), \Gamma(x_{i_1}, x_{i_2}), \ldots, \Gamma(x_{i_{k-1}}, x_{i_k})\}$ of constraints over P. The successive composition of constraints over P with respect to Γ is defined as

$$\bigotimes \Gamma(P) := \Gamma(x_{i_0}, x_{i_1}) \otimes \Gamma(x_{i_1}, x_{i_2}) \otimes \cdots \otimes \Gamma(x_{i_{k-1}}, x_{i_k}).$$

A concatenation of two paths $P_1 := (x_{i_0}x_{i_1}, \ldots, x_{i_{k-1}}x_{i_k})$ and $P_2 := (x_{i_k}x_{i_{k+1}}, \ldots, x_{i_{\ell-1}}x_{i_\ell})$, denoted by $P_1 + P_2$, is the path $(x_{i_0}x_{i_1}, \ldots, x_{i_{\ell-1}}x_{i_\ell})$. We write $\Pi(x, y, E)$ for the set of all paths from x to y on set E of edges.

Example 1. In Figure 1b, $P = (x_1x_4, x_4x_2, x_2x_3)$ is a path of length three in $\Pi(x_1, x_3, E)$, and $C = (x_2x_4, x_4x_5, x_5x_3, x_3x_2)$ is a cycle in $\Pi(x_2, x_2, E)$. The composition of the constraints over P, which is $\bigotimes \Gamma(P)$, is then $(x_1[-145, -96]x_4) \otimes (x_4[-15, -11]x_2) \otimes (x_2[-234, 21]x_3)$. Note that $(x_4[-15, -11]x_2)$ is equivalent to $(x_2[11, 15]x_4)$.

The following lemma states that the algebraic operations $\otimes$ and $\cap$ are sufficient to calculate the minimal network of an STN. More precisely, the minimal constraint between two variables x and y is equal to the intersection of the compositions of the constraints over the paths in $\Pi(x, y, E)$.

Lemma 1. *Let Γ be an STN and E be the set of edges of the constraint graph of Γ. Let Γ^{m} be the minimal network of Γ. Then for all $x, y \in V$ with $x \neq y$ we have*

$$\Gamma^{\mathrm{m}}(x, y) \;=\; \bigcap_{P \in \Pi(x,y,E)} \bigotimes \Gamma(P).$$

Proof. See [5, Section 3]. □

For constraint satisfaction problems chordal (or triangulated) constraint graphs have been identified as a class for which efficient algorithms exist [18].

Next, we provide a characterization of a chordal graph equivalent to that in [18], which is more suited for the purpose of the paper.

Definition 6. (Chordal Graph) A graph $G = (V, E)$ is said to be *chordal*, if for any edge $xy \in E$ and any path P from x to y on E with $|P| \geq 2$ there exists $z \in V$ with $xz, zy \in E$ such that $P = P_1 + P_2$, where P_1 is a path from x to z and P_2 a path from z to y.

Example 2. The graph in Figure 1a is chordal as it is complete. We can make a non-chordal graph chordal by triangulating it. For example, the graph in Figure 1b is not chordal, but we can make it chordal by adding edge x_3x_4 and obtain the graph in Figure 2b.

In the following lemma we characterize a refinement (of an STN) whose constraint graph is chordal. Such a refinement has a constraint network that is less dense than that of the minimal network while sharing some nice properties with the minimal network.

Definition 7. Given an STN Γ, a refinement $\Gamma^{\triangle}$ of Γ is said to be a *chordal minimal network* of Γ w.r.t. graph $G^{\triangle}$, if

1. $G^{\triangle} = (V, E^{\triangle})$ is the constraint graph of $\Gamma^{\triangle}$ and is chordal with $E^{\triangle} \supseteq E$;
2. $\Gamma^{\triangle}(x, y) = \Gamma^{\mathrm{m}}(x, y)$ for all $xy \in E^{\triangle}$.

Example 3. The minimal network Γ^{m} of Γ satisfies trivially the conditions for $\Gamma^{\triangle}$, as its constraint graph is complete and thus chordal. The STN in Figure 2b has a chordal constraint graph that contains the constraint graph of the STN in Figure 1b. Its constraints are minimal.

For two equivalent STNs Γ_1 and Γ_2 suppose $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ are their constraint graphs, respectively. If $G^{\triangle} = (V, E^{\triangle})$ is a chordal graph with $E^{\triangle} \supseteq E_1$ and $E^{\triangle} \supseteq E_2$, then the chordal minimal network $\Gamma^{\triangle}$ of Γ_1 w.r.t. $G^{\triangle}$ is also a chordal minimal network of Γ_2 w.r.t. $G^{\triangle}$, as Γ_1 and Γ_2 have the same minimal network.

The following characterization of $\Gamma^{\triangle}$ follows from its definition and Lemma 1.

Lemma 2. *Let Γ and $\Gamma^{\triangle}$, as well as E and $E^{\triangle}$, be specified as in Definition 7. Then for all $x, y \in V$ with $xy \in E^{\triangle}$ we have*

$$\Gamma^{\triangle}(x, y) \;=\; \bigcap_{P \in \Pi(x,y,E)} \bigotimes \Gamma(P).$$

3 REDUNDANCY IN SIMPLE TEMPORAL NETWORKS

In an STN some constraints can be redundant as they can be inferred from the rest of the constraints of the STN. In the remainder of this paper we assume that the STNs are consistent.

Definition 8. Let Γ be an STN. Then $\Gamma(x, y)$ is said to be *redundant* in Γ, if $\Gamma \setminus \{\Gamma(x, y)\} \models \Gamma(x, y)$. We say Γ is *prime*, if it does not contain any redundant constraints. A subset Γ' of Γ is called a *prime subnetwork* of Γ, if Γ' is prime and equivalent to Γ. The set of non-redundant constraints in Γ, denoted by Γ^{c}, is the *core* of Γ.

To obtain a prime subnetwork, a naive algorithm would first determine whether there exists a redundant constraint in Γ. If it does not find any, then Γ will be returned. Otherwise, for a constraint $\Gamma(x, y)$ that is redundant in Γ the algorithm sets $\Gamma \leftarrow \Gamma \setminus \{\Gamma(x, y)\}$ and repeats the procedure recursively.

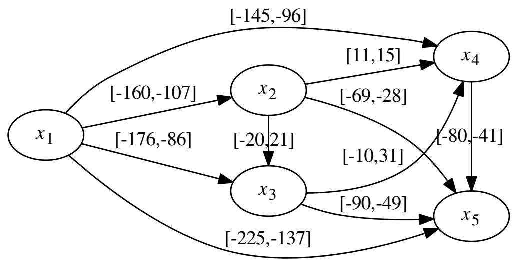
Determining the redundancy of a constraint $\Gamma(x, y)$ can be done by computing the minimal network $\tilde{\Gamma}^{\mathrm{m}}$ of $\tilde{\Gamma} := \Gamma \setminus \{\Gamma(x, y)\}$ and checking whether $\tilde{\Gamma}^{\mathrm{m}}(x, y)$ refines $\Gamma(x, y)$. Since there are $O(n^2)$ constraints in each of the $O(n^2)$ recursions, and since calculating a minimal network can be done in $O(n^3)$ (cf. [5]), the algorithm runs in $O(n^7)$. The algorithm can be improved to $O(n^5)$ by not revisiting constraints that were identified as non-redundant previously, because a non-redundant constraint in an STN Γ is also not redundant in any STN that is contained in Γ.

We observe that any prime subnetwork of Γ contains the core Γ^{c} as a subset of constraints. If the core Γ^{c} is a prime subnetwork of Γ, then it is the unique prime subnetwork. However, Γ^{c} is not necessarily a prime subnetwork of Γ, because removing a redundant constraint c of Γ can cancel the redundancy of another constraint, i.e. the latter constraint would remain redundant in Γ if the former constraint had not been removed. Thus, there can be more than one prime subnetworks of Γ and simultaneously removing all redundant constraints in Γ can lead to an STN that is not to equivalent to Γ, as the following simple example illustrates.

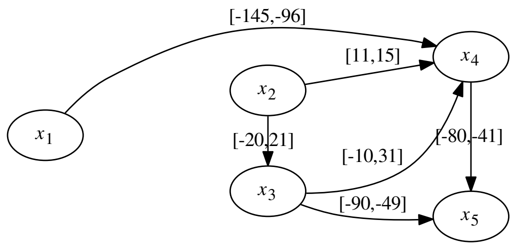
Example 4. Suppose that the STN consists of the constraints $(x\ [0, 1]\ y)$, $(x\ [2, 3]\ z)$, and $(y\ [2, 2]\ z)$, which translate to $0 \leq y - x \leq 1$, $2 \leq z - x \leq 3$, and $z - y = 2$ respectively. Then, it is easy to see that $(x\ [0, 1]\ y)$ and $(x\ [2, 3]\ z)$ are both redundant because $(x\ [0, 1]\ y) = (x\ [2, 3]\ z) \otimes (z\ [-2, -2]\ y)$ and $(x\ [2, 3]\ z) = (x\ [0, 1]\ y) \otimes (y\ [2, 2]\ z)$. However, removing both $(x\ [0, 1]\ y)$ and $(x\ [2, 3]\ z)$ will leave the single constraint $(y\ [2, 2]\ z)$ as the core, which obviously has a different solution set from the original STN.

For some other STNs the core is equivalent to the original STN.

Example 5. The STN in Figure 1b is the core of the STN in Figure 1a. Moreover, these two STNs are equivalent, because the minimal network of both of them is the STN shown in Figure 2a.



(a) The minimal network of the STNs from Figure 1a and from 1b.



(b) An STN whose constraint graph is chordal and contains the constraint graph of the STN from Figure 1b. It is equivalent to the STN from Figure 1b, and its constraints are minimal.

Figure 2. STNs equivalent to the STNs in Figure 1.

We observe that in Example 4, the presence of constraint $(y[2,2]z)$ makes the other two constraints dependent on each other with respect to their redundancy. Such STNs are *degenerated*.

Definition 9. Let Γ be an STN and $\Gamma^{\triangle}$ a chordal minimal network of Γ. A constraint $(x\ [a,b]\ y)$ is said to be *degenerated*, if $a = b$. STN Γ is said to be *degenerated* if there is $xy \in E^{\triangle}$ such that $\Gamma^{\triangle}(x,y) = (x\ [a,b]\ y)$ with $a = b$.

It is worth noting that if an STN is degenerated, say $(x\ [a,a]\ y)$, then we can easily fix it by either rewriting the constraint as $(x\ [a-\varepsilon, a+\varepsilon]\ y)$, or by removing constraints involving y and updating the constraints $\Gamma(x,z)$ using the rule $\Gamma(x,z) \cap (\Gamma(x,y) \otimes \Gamma(y,z))$.

In the following, we will show that for non-degenerated STNs, the core is the unique minimal subset of constraints that is equivalent to the original STN. To this end, we first characterize redundant constraints as the intersection of constraints over certain paths.

Lemma 3. *Let Γ be an STN and $G = (V,E)$ its constraint graph. Then for all $x,y \in V$ with $xy \in E$ the following are equivalent:*

(i) $\Gamma(x,y)$ *is redundant in* Γ*;*

(ii) $\Gamma(x,y) \supseteq \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P)$.

Proof. (i) $\Rightarrow$ (ii). Let $\Gamma_0 := \Gamma \setminus \{\Gamma(x,y)\}$. Then, since $\Gamma(x,y)$ is redundant in Γ we have $\Gamma_0 \models \Gamma(x,y)$ thus $(\Gamma_0)^{\mathrm{m}} \models \Gamma(x,y)$, where $(\Gamma_0)^{\mathrm{m}}$ is the minimal network of Γ_0. Hence

$$\Gamma(x,y) \supseteq (\Gamma_0)^{\mathrm{m}}(x,y). \tag{2}$$

By Lemma 1 we know that

$$(\Gamma_0)^{\mathrm{m}}(x,y) = \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P). \tag{3}$$

From (2) and (3) it follows that

$$\Gamma(x,y) \supseteq \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P).$$

(ii) $\Rightarrow$ (i). For all paths between x and y on $E \setminus \{xy\}$, we have $\Gamma \models \Gamma(P)$ and thus $\Gamma \models \bigotimes \Gamma(P)$. Therefore

$$\Gamma \models \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P).$$

By applying our assumption (ii) we then have $\Gamma \models \Gamma(x,y)$, and we showed (i). □

Example 6. The constraint $\Gamma(x_3,x_4) = (x_3[-201,72]x_4)$ in the STN Γ from Figure 1a is redundant in Γ, because $(x_3[-201,72]x_4) \supseteq (x_3[-10,31]x_4) = \Gamma(x_3,x_2) \otimes \Gamma(x_2,x_4) \cap \Gamma(x_3,x_5) \otimes \Gamma(x_5,x_4) \supseteq \bigcap_{P\in\Pi(x_3,x_4,E\setminus\{x_3x_4\})} \bigotimes \Gamma(P)$.

As noted earlier $\Gamma^{\triangle}$ shares some nice properties with the minimal network of Γ, thus allowing a characterization of the redundant constraints in $\Gamma^{\triangle}$.

Proposition 4. *Let Γ and $\Gamma^{\triangle}$ as well as E and $E^{\triangle}$ be specified as in Definition 7. Then for all $x,y \in V$ with $xy \in E^{\triangle}$ the following are equivalent:*

(i) $\Gamma^{\triangle}(x,y)$ *is redundant in* $\Gamma^{\triangle}$

(ii) $\Gamma^{\triangle}(x,y) = \bigcap_{P\in\Pi(x,y,E^{\triangle}\setminus\{xy\})} \bigotimes \Gamma^{\triangle}(P)$

(iii) $\Gamma^{\triangle}(x,y) = \bigcap_{xz,zy\in E^{\triangle}} \Gamma^{\triangle}(x,z) \otimes \Gamma^{\triangle}(z,y)$

Proof. See Appendix A. □

Example 7. Consider the minimal network in Figure 2a as $\Gamma^{\triangle}$ for Γ in Figure 1a. Then we have $\Gamma^{\triangle}(x_3,x_4) = (x_3[-10,31]x_4) = (\Gamma^{\triangle}(x_3,x_1) \otimes \Gamma^{\triangle}(x_1,x_4)) \cap (\Gamma^{\triangle}(x_3,x_2) \otimes \Gamma^{\triangle}(x_2,x_4)) \cap (\Gamma^{\triangle}(x_3,x_5) \otimes \Gamma^{\triangle}(x_5,x_4))$.

There is a connection between redundant constraints in Γ and in $\Gamma^{\triangle}$.

Proposition 5. *Let Γ and $\Gamma^{\triangle}$ as well as E and $E^{\triangle}$ be specified as in Definition 7. Furthermore, let Γ be not degenerated. Then for all $x,y \in V$ with $xy \in E^{\triangle}$ the following are equivalent:*

(i) $\Gamma(x,y)$ *is redundant in* Γ

(ii) $\Gamma^{\triangle}(x,y)$ *is redundant in* $\Gamma^{\triangle}$

(iii) $\Gamma^{\triangle}(x,y) = \bigcap_{\substack{P\in\Pi(x,y,E)\\ |P|\geq 2}} \bigotimes \Gamma(P)$

Proof. See Appendix A. □

Example 8. Taking the STN Γ in Figure 1a and the STN $\Gamma^{\triangle}$ in Figure 2a as an example of the above proposition, we can see that the constraint $\Gamma(x_3, x_4) = (x_3[-201, 72]x_4)$ is redundant in Γ while $\Gamma^{\triangle}(x_3, x_4) = (x_3[-10, 31]x_4)$ is redundant in $\Gamma^{\triangle}$.

The following theorem affirms that for a non-degenerated STN the core resulting from removing redundant constraint simultaneously is the unique prime subnetwork of the STN.

Theorem 6. *Let Γ be a non-degenerated STN. Then the core Γ^c of Γ is equivalent to Γ and the unique prime subnetwork of Γ.*

Proof. The proof is similar to that for RCC8 constraints (see [9, Theorem 29]). Suppose $r_1, r_2, \ldots, r_n$ are the redundant constraints of Γ. Let $\Gamma_0 := \Gamma$ and $\Gamma_i := \Gamma_{i-1} \setminus \{r_i\}$ for $1 \leq i \leq n$. Note that Γ_n is precisely Γ^c, the core of Γ.

We prove the claim by contradiction. Suppose k with $k < n$ is the largest integer such that $\Gamma_k \equiv \Gamma$. Let $r_{k+1} = \Gamma(x, y)$, thus $\Gamma(x, y)$ is redundant in Γ. Further, let $\Gamma^{\triangle}$ be a chordal minimal network of Γ. Then, on the one hand, we know by Proposition 5 that $\Gamma^{\triangle}(x, y)$ is redundant in $\Gamma^{\triangle}$. On the other hand, since $\Gamma^{\triangle}$ is also a chordal minimal network of Γ_k, by Proposition 5 we also have that $\Gamma(x, y)$ is redundant in Γ_k. Therefore, $\Gamma_{k+1} = \Gamma_k \setminus \{r_{k+1}\} \equiv \Gamma_k \equiv \Gamma$, contradicting our assumption that $k < n$ is the largest integer such that Γ_k is equivalent to Γ. Therefore, $k = n$ and Γ^c is equivalent to Γ, thus it is a prime subnetwork of Γ that is uniquely defined. □

With this result, we can use the core as a simplification of the original STN and obtain a maximal reduction in the size of representation, without changing the semantic essence.

4 EFFICIENT ALGORITHM FOR CALCULATING THE CORE

The result below follows directly from Propositions 5 and 4, and allows for efficient identification of redundant constraints in STNs.

Proposition 7. *Let Γ and $\Gamma^{\triangle}$ as well as E and $E^{\triangle}$ be specified as in Definition 7. Furthermore, let Γ be not degenerated. Then, for all $x, y \in V$ with $xy \in E^{\triangle}$ the following are equivalent:*

(i) $\Gamma(x, y)$ *is redundant in* Γ*;*

(ii) $\Gamma^{\triangle}(x, y) = \bigcap_{xz, zy \in E^{\triangle}} \Gamma^{\triangle}(x, z) \otimes \Gamma^{\triangle}(z, y)$.

With the aid of Proposition 7, we propose Algorithm 1 to efficiently calculate the core of a non-degenerated STN Γ, which is also the prime subnetwork of Γ by Theorem 6. In this algorithm, we first construct a chordal graph $G^{\triangle} = (V, E^{\triangle})$ by triangulating the constraint graph $G = (V, E)$ of Γ (line 3). Then, we construct the chordal minimal network $\Gamma^{\triangle}$ w.r.t. $G^{\triangle}$ (line 4). After that, we iteratively retrieve the core of Γ edge by edge (lines 5–11). By Proposition 7, to identify the redundancy of a constraint involving an edge $xy \in E$, we only need to check if the constraint corresponding to edge xy in $E^{\triangle}$ equals the intersection of the compositions of all paths of length two from x to y in $E^{\triangle}$ (lines 8–11).

Example 9. To calculate the core of the STN Γ in Figure 1a, Algorithm 1 first triangulates the constraint graph $G = (V, E)$ of Γ. In this case, as G is complete, any triangulation of G is still G. Then, by calculating the minimal network, $\Gamma^{\triangle}$ is obtained, which is the STN shown in Figure 2a. For each edge $x_i x_j$ in E, the algorithm checks if the constraint $\Gamma^{\triangle}(x_i, x_j)$ coincides with the intersection

Algorithm 1: CORE(Γ)

```
Input  : A non-degenerated STN $\Gamma$.
Output: $\Gamma^c$, the core of $\Gamma$.
1  $\Gamma^c \leftarrow \emptyset$ ;
2  $G = (V, E) \leftarrow$ the constraint graph of $\Gamma$;
3  $G^{\triangle} = (V, E^{\triangle}) \leftarrow$ TRIANGULATE($G$);
4  $\Gamma^{\triangle} \leftarrow$ CHORDALMINIMALNETWORK($\Gamma, G^{\triangle}$);
5  while $E \neq \emptyset$ do
6      $xy \leftarrow E$.POP() ;
7      $I \leftarrow (x\,[-\infty, \infty]\,y)$ ;
8      foreach $z$ with $(x, z), (z, y) \in E^{\triangle}$ do
9          $I \leftarrow I \cap (\Gamma^{\triangle}(x, z) \otimes \Gamma^{\triangle}(z, y))$ ;
10     if $\Gamma^{\triangle}(x, y) \neq I$ then
11         $\Gamma^c \leftarrow \Gamma^c \cup \Gamma(x, y)$;
12 return $\Gamma^c$.
```

$\bigcap_{x_i x_k, x_k x_j \in E^{\triangle}} \Gamma^{\triangle}(x_i, x_k) \otimes \Gamma^{\triangle}(x_k, x_j)$. The result shows that $\Gamma^{\triangle}(x_1, x_2), \Gamma^{\triangle}(x_1, x_3), \Gamma^{\triangle}(x_1, x_5), \Gamma^{\triangle}(x_2, x_5)$, and $\Gamma^{\triangle}(x_3, x_4)$ are such constraints. Therefore, we know that the corresponding constraints in Γ are redundant, and we obtain the core Γ^c in Figure 1b by removing them.

In what follows, we analyze the time complexity of Algorithm 1 in terms of the number of triangles t in the graph $G^{\triangle}$. Note that a sparsely structured graph can contain much fewer triangles than a complete graph of the same number of vertices. For each execution of the while loop (lines 5–13) the algorithm checks the triangles in $G^{\triangle}$ that contain xy as an edge. Hence, the while loop checks each triangle in $G^{\triangle}$ at most three times and, thus, runs in $O(t + |E|)$ time. To obtain a sparse chordal graph $G^{\triangle}$, we can use the *maximum cardinality search* algorithm [19] and a simple *fill-in* procedure [12]. In particular, the maximum cardinality search algorithm visits the vertices of G in an order such that, at any point, a vertex is visited that has the largest number of visited neighbours. Consequently, a vertex ordering α is produced. Then, the fill-in procedure considers the vertices in G one by one with respect to the vertex ordering α, and connects each pair of vertices in the neighbourhood of the vertex at hand with an undirected *fill* edge (if an edge is not already present). The obtained graph $G^{\triangle}$ is then a triangulation of G [7]. The maximum cardinality search algorithm has the nice property that it does not lead to any fill edges if the graph is already chordal. The entire operation of triangulating G in the aforementioned manner is linear in the size of $G^{\triangle}$, viz., $O(|V| + |E^{\triangle}|)$. Further, to construct $\Gamma^{\triangle}$ we can use the state-of-the-art algorithm P^3C [13], which runs in $O(t)$ time. Therefore, Algorithm 1 runs in $O(t + |V| + |E^{\triangle}|)$ time. We note that the running time of Algorithm 1 depends on the structure of the constraint graphs: for complete graphs an upper bound is $O(n^3)$; however, if the constraint graph is a tree, then Algorithm 1 runs in $O(n)$. For determining the prime subnetworks, this is a significant improvement over a naive algorithm that runs in $O(n^5)$ (cf. Section 3).

5 EVALUATION

In this section, we evaluate the previous theoretical results using a large benchmark dataset comprising over a thousand of STNs of various nature. Note that by the analysis that took place in the previous section, the performance of Algorithm 1 is strongly dependent on the P^3C algorithm for constructing $G^{\triangle}$, which has been shown to be

Table 1. Results on redundancy reduction

dataset	#STNs	#deg.[a]	properties[b]			reduction[c]			
			$\lvert\mathbf{V}\rvert$	$\lvert\mathbf{E}\rvert$	$\overline{\mathbf{D}}$	$\boldsymbol{\mu}$	**min**	**max**	$\boldsymbol{\sigma}$
Chordal-1	250	18	1 000	75 840–499 490	0.532	97.43%	94.41%	98.99%	1.25
Chordal-2	130	11	214–3 125	22 788–637 009	0.509	96.92%	94.61%	98.27%	0.98
Scale-free-1	130	48	1 000	1 996–67 360	0.039	10.27%	0.10%	55.23%	13.09
Scale-free-2	160	18	250–1 000	2 176–3 330	0.025	3.30%	0.37%	17.93%	3.74
New York	170	0	108–3 906	113–6 422	0.006	0.01%	0.00%	0.19%	0.04
Diamonds	130	0	111–2 751	111–2 751	0.006	0.00%	0.00%	0.00%	0.00
Job-shop	400	5	17–1 321	32–110 220	0.142	49.45%	0.00%	73.69%	16.91
HTN	121	0	500–625	748–1 599	0.007	0.02%	0.00%	0.15%	0.04

[a] The number of degenerated STNs.
[b] $|\mathbf{V}|$,$|\mathbf{E}|$, and $\overline{\mathbf{D}}$: the number of vertices, the number of edges, and the average density of the constraint graphs of STNs respectively.
[c] $\boldsymbol{\mu}$, **min**, **max**, and $\boldsymbol{\sigma}$: the average, the minimal, the maximal, and the standard deviation value of reduction rate respectively.

very efficient in [13]. In light of that, and as we are more interested in obtaining results on redundancy reduction for STNs, we will not report on the computing time of our implementation.

Regarding the benchmark dataset, we employed the dataset of 1491 STNs used in the work of Planken et al. in [14]. The basic properties of that dataset are presented in Table 1. More details about the various natures of the dataset can be found in [14, Section 4]. The STNs vary from random scale-free networks and parts of the road network of New York City, to STNs generated from hierarchical task networks (HTNs) and job-shop scheduling problems. Note that Chordal-1 and Scale-free-1 contain STNs of a fixed number of variables, while Chordal-2 and Scale-free-2 contain STNs of a variant number of variables. It is worth noting that a small percentage of STNs are degenerated. To be exact, $100/1491$ of the STNs are degenerated. The number of degenerated STNs varies per nature of dataset. For example, only 5 out of the 400 STNs of the job-shop scheduling problems dataset (Job-shop) are degenerated. On the other hand, over a third of the STNs of the 1000-variable scale-free STNs dataset are degenerated. In any case, all degenerated STNs were easily and minimally repaired by introducing a small weight ε to each degenerated interval $[a, a]$, hence, modifying each such interval to $[a-\varepsilon, a+\varepsilon]$. We also observe that Chordal-1 and Chordal-2, which are STNs whose constraint graphs are chordal, are the densest ones, followed by the STNs of the job-shop scheduling problems dataset, viz., Job-shop, and the scale-free networks datasets, viz., Scale-free-1 and Scale-free-2. On the contrary, STNs derived from the road network of New York City (New York), HTNs (HTN), and diamond-shaped networks (Diamonds) are extremely sparse, to the point of resembling trees, and are thus almost devoid of redundancy (as we will see in the evaluation to follow).

Regarding the results of our evaluation, Table 1 provides a detailed description of redundancy reduction per nature of dataset. Given the fact that chordal STNs are the densest ones, it is not surprising to obtain reduction rates lying between 94.41% to 98.99%, with a reduction rate of about 97% on average. With respect to the job-shop scheduling problems, we obtained an average reduction rate of 49.45%, while the reduction rate among the STNs reaches as high as 73.69%. Further, we obtain a 10.27% average reduction rate for scale-free STNs of a fixed number of variables, and a 3.3% average reduction rate for scale-free STNs of a variant number of variables. Similarly to the case of the job-scheduling problems, some of the STNs have less redundant constraints because they are already sparse. Finally, as expected due to their extreme sparseness, no significant reduction rate is obtained for any of the STNs derived from the road network of New York City, the HTNs, or the diamond-shaped networks. For instance, all the constraints in the dataset of diamond-shaped networks are non-redundant.

As we strongly hinted earlier, the density of the constraint graph of a given STN correlates with the reduction that we can achieve in the number of constraints of that STN. The question arises of how big the correlation between densities and reduction rates really is, at least with respect to the dataset at hand. To answer this question, we use the *Spearman's rank correlation coefficient* [21], a non-parametric test that is used to measure the degree of association between two variables. It assesses how well the relationship between two variables x and y can be described using a monotonic function. Like other correlation coefficients, this one varies between -1 and $+1$, with 0 implying no correlation. Correlations of -1 or $+1$ imply an exact monotonic relationship. Positive correlations imply that as x increases, so does y. Negative correlations imply that as x increases, y decreases. By using the average density $\overline{D}$ and the average reduction rate μ as our variables, along with their respective values as provided in Table 1, we can obtain a Spearman's rank correlation coefficient of 0.99, which demonstrates that each of our two variables is almost a perfect monotone function of the other. This result suggests that a strong correlation between densities and reduction rates exists, despite the various natures of the benchmark dataset.

6 CONCLUSION AND FUTURE WORK

In this paper, we investigated the redundancy problem of STNs. In particular, we showed that for any non-degenerated STN Γ, there is a unique minimal subset Γ^c of its constraints (i.e. the core of Γ) such that all of the constraints in Γ^c are not redundant in Γ, and Γ^c has the same set of solutions as Γ. We proposed an efficient algorithm to calculate the core, which runs in time linear in the number of triangles in a chordal graph that results from triangulating the constraint graph of Γ. Our experiments with a large benchmark dataset of STNs unveiled a vast number of redundant constraints, which suggests that the cores of the STNs exhibit a significant reduction in redundancy in practice. With respect to our dataset, we also identified a strong correlation between densities and reduction rates. For future work, it would be interesting to investigate whether that statistical dependence between our measures will hold if more datasets of various nature were to be considered.

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A PROOFS

Proposition 4 . *Let Γ and $\Gamma^\triangle$ as well as E and $E^\triangle$ be specified as in Definition 7. Then for all $x, y \in V$ with $xy \in E^\triangle$ the following are equivalent:*

(i) $\Gamma^\triangle(x,y)$ *is redundant in* $\Gamma^\triangle$
(ii) $\Gamma^\triangle(x,y) = \bigcap_{P\in\Pi(x,y,E^\triangle\setminus\{xy\})} \bigotimes\Gamma^\triangle(P)$
(iii) $\Gamma^\triangle(x,y) = \bigcap_{xz,zy\in E^\triangle} \Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)$

Proof. (i) ⇒ (ii). Suppose $\Gamma^\triangle(x,y)$ is redundant in $\Gamma^\triangle$. Then, by applying Lemma 3, we know that

$$\Gamma^\triangle(x,y) \supseteq \bigcap_{P\in\Pi(x,y,E^\triangle\setminus\{xy\})} \bigotimes\Gamma^\triangle(P).$$

On the other hand, $\Gamma^\triangle(x,y)$ is minimal in $\Gamma^\triangle$. Therefore

$$\Gamma^\triangle(x,y) \subseteq \bigcap_{P\in\Pi(x,y,E^\triangle\setminus\{xy\})} \bigotimes\Gamma^\triangle(P).$$

and we showed (ii).

(ii) ⇒ (iii). Let P be a path between x and y on $E^\triangle\setminus\{xy\}$. Then, because $G^\triangle$ is chordal, there exists a $z'\in V$ with $(x,z'),(z',y)\in E^\triangle$, such that P is a concatenation of paths P_1 and P_2, where P_1 is between x and z' and P_2 is between z' and y. Since $\Gamma^\triangle(x,z')$ and $\Gamma^\triangle(z',y)$ are minimal in $\Gamma^\triangle$, we have

$$\bigotimes\Gamma^\triangle(P_1) \supseteq \Gamma^\triangle(x,z')$$

and

$$\bigotimes\Gamma^\triangle(P_2) \supseteq \Gamma^\triangle(z',y).$$

Hence,

$$\begin{aligned}\bigotimes\Gamma^\triangle(P) &= \bigotimes\Gamma^\triangle(P_1)\otimes\bigotimes\Gamma^\triangle(P_2)\\ &\supseteq \Gamma^\triangle(x,z')\otimes\Gamma^\triangle(z',y)\\ &\supseteq \bigcap_{xz,zy\in E^\triangle}\Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y).\end{aligned}$$

Because the preceding statement holds for any path P between x and y on $E^\triangle\setminus\{xy\}$, we have

$$\bigcap_{P\in\Pi(x,y,E^\triangle\setminus\{xy\})}\bigotimes\Gamma^\triangle(P) \supseteq \bigcap_{xz,zy\in E^\triangle}\Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)$$

and by applying our assumption (ii), we have

$$\Gamma^\triangle(x,y) \supseteq \bigcap_{xz,zy\in E^\triangle}\Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y).$$

The other inclusion

$$\Gamma^\triangle(x,y) \subseteq \bigcap_{xz,zy\in E^\triangle}\Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)$$

follows from the minimality of $\Gamma^\triangle(x,y)$ in $\Gamma^\triangle$. Thus we showed (iii).

(iii) ⇒ (i). We first observe that for all $x,y,z\in V$ with $xy,xz,zy\in E^\triangle$

$$\Gamma^\triangle\setminus\{\Gamma^\triangle(x,y)\} \models \{\Gamma^\triangle(x,z),\Gamma^\triangle(z,y)\}$$

and

$$\{\Gamma^\triangle(x,z),\Gamma^\triangle(z,y)\} \models \Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)$$

and therefore

$$\Gamma^\triangle\setminus\{\Gamma^\triangle(x,y)\} \models \Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y).$$

Thus we have

$$\Gamma^\triangle\setminus\{\Gamma^\triangle(x,y)\} \models \bigcap_{xz,zy\in E^\triangle}\Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)$$

By applying our assumption (iii) we then have

$$\Gamma^\triangle\setminus\{\Gamma^\triangle(x,y)\} \models \Gamma^\triangle(x,y)$$

and we showed (i). □

Non-degenerated STNs have the following property.

Lemma 8. *Let Γ and $\Gamma^\triangle$ as well as E and $E^\triangle$ be specified as in Definition 7. Suppose Γ is not degenerated. Then there is a fixed $\varepsilon>0$ such that for any cycle $C\in\Pi(x,x,E)$ we have that*

$$\bigotimes\Gamma(C) \supseteq (x\,[-\varepsilon,\varepsilon]\,x)$$

Proof. We first note that there exists an $\varepsilon>0$ such that for any $x\in V$ and for any path $P\in\Pi(x,x,E^\triangle)$ of length 3, i.e. P is a triangle, we have $\bigotimes\Gamma^\triangle(P)\supseteq(x\,[-\varepsilon,\varepsilon]\,x)$. This is because there are only finitely many triangles in $E^\triangle$ and the composition of non-degenerated constraints is again non-degenerate. Then, because $G^\triangle$ is chordal, for any cycle $C\in\Pi(x,x,E)$ with $xy\in C$ there exists a $z\in V$ with $xz,zy\in E^\triangle$ such that $C=P_1+P_2+yx$ with $P_1\in\Pi(x,z,E)$ and $P_2\in\Pi(z,y,E)$. Then

$$\begin{aligned}\bigotimes\Gamma(C) &= \bigotimes\Gamma(P_1)\otimes\bigotimes\Gamma(P_2)\otimes\Gamma(y,x)\\ &\supseteq \bigotimes\Gamma^\triangle(P_1)\otimes\bigotimes\Gamma^\triangle(P_2)\otimes\Gamma^\triangle(y,x)\\ &\supseteq \Gamma^\triangle(x,z)\otimes\Gamma^\triangle(z,y)\otimes\Gamma^\triangle(y,x)\\ &\supseteq (x\,[-\varepsilon,\varepsilon]\,x)\end{aligned}$$

□

Example 10. The STN Γ_2 in Figure 2b is not degenerated. Consider the paths $C_1=(x_2x_4)$, $C_2=(x_4x_3,x_3x_5,x_5x_4)$, $C_3=(x_4x_3,x_3x_2)$. Note that $C=C_1+C_2+C_3$ and C_2 are cycles. Then $\bigotimes\Gamma_2(C)=\bigotimes\Gamma_2(C_1)\otimes\bigotimes\Gamma_2(C_2)\otimes\bigotimes\Gamma_2(C_3)=(x_2[11,15]x_4)\otimes(x_4[-80,41]x_4)\otimes(x_4[-52,30]x_2)=(x_2[-121,86]x_2)\supseteq(x_2[-41,45]x_2)=\bigotimes\Gamma_2(C_1)\otimes\bigotimes\Gamma_2(C_3)\supseteq(x_2[-\varepsilon_C,\varepsilon_C]x_2)$, where we can choose $\varepsilon_C=41$.

Proposition 5 . *Let Γ and $\Gamma^\triangle$ as well as E and $E^\triangle$ be specified as in Definition 7. Furthermore, Γ is not degenerated. Then for all $x,y\in V$ with $xy\in E^\triangle$ the following are equivalent:*

(i) $\Gamma(x,y)$ *is redundant in* Γ
(ii) $\Gamma^\triangle(x,y)$ *is redundant in* $\Gamma^\triangle$

(iii) $\Gamma^{\triangle}(x,y) \;=\; \bigcap_{\substack{P\in\Pi(x,y,E)\\|P|\geq 2}} \bigotimes \Gamma(P)$

Proof. (i) $\Rightarrow$ (ii). Because $\Gamma^{\triangle} \models \Gamma$, we have

$$\Gamma^{\triangle} \setminus \{\Gamma^{\triangle}(x,y)\} \;\models\; \Gamma \setminus \{\Gamma(x,y)\},$$

and since, by our assumption, $\Gamma \setminus \{\Gamma(x,y)\} \equiv \Gamma$, we have

$$\Gamma^{\triangle} \setminus \{\Gamma^{\triangle}(x,y)\} \;\models\; \Gamma.$$

Note that, $\Gamma^{\mathrm{m}} \;\models\; \Gamma^{\triangle}$ and $\Gamma^{\triangle} \;\models\; \Gamma$. Then, because $\Gamma^{\mathrm{m}} \equiv \Gamma$, we also have $\Gamma \;\models\; \Gamma^{\triangle}$. Therefore $\Gamma \equiv \Gamma^{\triangle}$ and we finally have

$$\Gamma^{\triangle} \setminus \{\Gamma^{\triangle}(x,y)\} \;\models\; \Gamma^{\triangle},$$

and showed (ii).

(ii) $\Rightarrow$ (iii). Suppose $\Gamma^{\triangle}(x,y)$ is redundant in $\Gamma^{\triangle}$. Then, by Proposition 4 we know that

$$\Gamma^{\triangle}(x,y) \;=\; \bigcap_{xz,zy\in E^{\triangle}} \Gamma^{\triangle}(x,z) \otimes \Gamma^{\triangle}(z,y).$$

By applying Lemma 2 we then have

$$\Gamma^{\triangle}(x,y)$$
$$= \bigcap_{xz,zy\in E^{\triangle}} \left(\bigcap_{P_1\in\Pi(x,z,E)} \bigotimes \Gamma(P_1) \right) \otimes \left(\bigcap_{P_2\in\Pi(z,y,E)} \bigotimes \Gamma(P_2) \right)$$

Since STP constraints are distributive (composition distributes over intersections) we have

$$\Gamma^{\triangle}(x,y) \;=\; \bigcap_{\substack{xz,zy\in E^{\triangle}\\P_1\in\Pi(x,z,E)\\P_2\in\Pi(z,y,E)}} \bigotimes \Gamma(P_1+P_2) \qquad (4)$$

On the other hand, because $G^{\triangle}$ is chordal, P is a path between x and y on E with $|P| \geq 2$ if and only if there exists a $z \in V$ with $(x,z),(z,y) \in E^{\triangle}, z \neq x, z \neq y$, such that P is a concatenation of paths P_1 and P_2, where P_1 is between x and z and P_2 is between z and y. Hence we have

$$\bigcap_{\substack{P\in\Pi(x,y,E)\\|P|\geq 2}} \bigotimes \Gamma(P) \;=\; \bigcap_{\substack{xz,zy\in E^{\triangle}\\P_1\in\Pi(x,z,E)\\P_2\in\Pi(z,y,E)}} \bigotimes \Gamma(P_1+P_2) \qquad (5)$$

From (4) and (5) it follows

$$\Gamma^{\triangle}(x,y) \;=\; \bigcap_{\substack{P\in\Pi(x,y,E)\\|P|\geq 2}} \bigotimes \Gamma(P)$$

and we showed (iii).

(iii) $\Rightarrow$ (i): Suppose

$$\Gamma^{\triangle}(x,y) \;=\; \bigcap_{\substack{P\in\Pi(x,y,E)\\|P|\geq 2}} \bigotimes \Gamma(P)$$

We first note that we can partition the set of paths between x and y on E with $|P| \geq 2$ into two subsets $\mathcal{P}_1$ and $\mathcal{P}_2$, where

$$\mathcal{P}_1 := \{P \in \Pi(x,y,E) \mid xy \in P, |P| \geq 2\},$$

$$\mathcal{P}_2 := \{P \in \Pi(x,y,E) \mid xy \notin P, |P| \geq 2\}.$$

Hence

$$\Gamma^{\triangle}(x,y) \;=\; \bigcap_{P\in\mathcal{P}_1\dot{\cup}\mathcal{P}_2} \bigotimes \Gamma(P). \qquad (6)$$

We note that $\mathcal{P}_1$ consists of paths that are of the form $C_1 + (xy)$, $(xy) + C_2$, or $C_1 + (xy) + C_2$, where C_1 and C_2 are cycles on E with $|C_1| \geq 2$ and $|C_2| \geq 2$. Then, since Γ is not degenerated, by Lemma 8, there is an $\varepsilon > 0$ such that $\bigotimes \Gamma(C_1) \;\supseteq\; (x\,[-\varepsilon,\varepsilon]\,x)$, $\bigotimes \Gamma(C_2) \;\supseteq\; (y\,[-\varepsilon,\varepsilon]\,y)$, and we have, without loss of generality, for $P = C_1 + \{xy\} + C_2$

$$\begin{aligned}\bigotimes \Gamma(P) &= \bigotimes \Gamma(C_1 + \{xy\} + C_2)\\ &= \bigotimes \Gamma(C_1) \,\otimes\, \Gamma(x,y) \,\otimes\, \bigotimes \Gamma(C_2)\\ &\supseteq (x\,[-\varepsilon,\varepsilon]\,x) \otimes \Gamma(x,y)\\ &= (x\,[a-\varepsilon,b+\varepsilon]\,y),\end{aligned}$$

where $\Gamma(x,y) = (x\,[a,b]\,y)$. Hence,

$$\bigcap_{P\in\mathcal{P}_1} \bigotimes \Gamma(P) \;\supseteq\; (x\,[a-\varepsilon,b+\varepsilon]\,y). \qquad (7)$$

Thus by (6) and (7) we have

$$\begin{aligned}\Gamma^{\triangle}(x,y) &= \bigcap_{P\in\mathcal{P}_1\dot{\cup}\mathcal{P}_2} \bigotimes \Gamma(P)\\ &= \bigcap_{P\in\mathcal{P}_1} \bigotimes \Gamma(P) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)\\ &\supseteq (x\,[a-\varepsilon,b+\varepsilon]\,y) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)\\ &\supseteq (x\,[a,b]\,y) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)\\ &= (x\,[a,b]\,y) \cap \bigcap_{P\in\mathcal{P}_1} \bigotimes \Gamma(P) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)\\ &= \Gamma(x,y) \,\cap \bigcap_{P\in\mathcal{P}_1\dot{\cup}\mathcal{P}_2} \bigotimes \Gamma(P)\\ &= \Gamma(x,y) \,\cap\, \Gamma^{\triangle}(x,y)\\ &= \Gamma^{\triangle}(x,y)\end{aligned}$$

Consequently we have

$$\begin{aligned}&(x\,[a-\varepsilon,b+\varepsilon]\,y) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)\\ &\quad= (x\,[a,b]\,y) \cap \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P).\end{aligned}$$

By setting $(x\,[c,d]\,y) := \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P)$ we then have

$$(x\,[a-\varepsilon,b+\varepsilon]\,y) \cap (x\,[c,d]\,y) = (x\,[a,b]\,y) \cap (x\,[c,d]\,y),$$

which yields $[\max(a-\varepsilon,c), \min(b+\varepsilon,d)] = [\max(a,c), \min(b,d)]$ which holds only if $a \leq c$ and $b \geq d$, i.e. $(x\,[a,b]\,y) \supseteq (x\,[c,d]\,y)$. Therefore

$$\Gamma(x,y) \supseteq \bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P).$$

Since

$$\bigcap_{P\in\mathcal{P}_2} \bigotimes \Gamma(P) = \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P)$$

we have

$$\Gamma(x,y) \supseteq \bigcap_{P\in\Pi(x,y,E\setminus\{xy\})} \bigotimes \Gamma(P)$$

and by Lemma 3 we have that $\Gamma(x,y)$ is redundant in Γ. $\square$

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