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Can a Condorcet Rule Have a Low Coalitional Manipulability?

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Abstract. We investigate the possibility of designing a voting rule that both meets the Condorcet criterion and has a low vulnerability to coalitional manipulation. For this purpose, we examine the *Condorcification* of a voting rule, where the original rule is altered to elect the Condorcet winner when one exists, and we study its impact on coalitional manipulability. A recent result states that for a large class of voting rules based on strict total orders, their Condorcification is at most as coalitionally manipulable as the original rule. We show that for most of them, the improvement is strict. We extend these results to a broader framework that includes weak orders and cardinal voting rules. These results support the main message of this paper: when searching for a "reasonable" voting rule with minimal coalitional manipulability, investigations can be restricted to Condorcet rules. In other words, in a class of "reasonnable" voting rules, it is possible to have both the Condorcet criterion and a minimal vulnerability to coalitional manipulation.

1 Introduction

Any non-dictatorial voting rule with three eligible candidates or more[3] is vulnerable to manipulation by a single manipulator, who may secure an outcome that she prefers to the result of sincere voting by misrepresenting her preferences [12, 32]. Although this result is frequently cited under the form of Gibbard-Satterthwaite theorem, which deals only with *ordinal* voting rules (i.e. whose ballots are orders of preferences), Gibbard's fundamental theorem applies to any *game form*, where available strategies may be objects of any kind, for example grades [12].

Once this negative result is known, a possible direction consists in trying to mitigate the impact of manipulation. For example, one can investigate to what extent classic voting rules are manipulable, and try to identify ways of designing less manipulable voting rules.

In the case of manipulation by a single voter, assuming a "reasonable" voting rule and a large electorate (like in most political elections), it is unlikely that a voter is pivotal, which is both supported by theory [25, 24, 10, 11, 33] and analysis of real-life elections.

In this paper, we rather focus on coalitional manipulability, where a coalition of voters, by misrepresenting their preferences, may secure an outcome that they all prefer to the result of sincere voting. The very existence of this type of manipulation can have a strong practical impact on voting, even if all voters choose to vote sincerely. Indeed, whereas implementing a full-scale manipulation can be difficult, sincere voters may find out *a posteriori* that a coalitional manipulation was possible. This happened during the 2002 Presidential election in France, where left-wing voters discovered only after the election that a concerted ballot in favor of the main leftist candidate would probably have avoided the election of the main rightist candidate [5]. Such a scenario may result in a feeling of regret about the ballots cast, questions about the legitimacy of the outcome, and doubts about the voting rule itself, since for some of the voters, sincere ballots did not defend best their opinions.

To quantify the degree of coalitional manipulability of a voting rule, several indicators have been defined [19, 31, 21, 34, 8, 28, 36, 30]. One of the most studied is the *coalitional manipulability rate*, which is the probability that the voting rule is coalitionally manipulable (CM) in a random profile of preferences, under a given assumption on the probabilistic structure of the population (called *culture*).

In this paper, we consider a more detailed indicator: the set of profiles in which a given voting rule is CM [20]. It is closely related to the coalitional manipulability rate: a voting rule f is less CM than another rule g in this sense of set inclusion if and only if, in any culture, f has a lower coalitional manipulability rate than g.

Several authors have used a theoretical approach [19, 20, 17, 21, 16, 9, 22, 8, 29, 18], computer simulations [19, 29, 30, 13, 14, 15] or experimental results [4, 36, 14, 15] to evaluate the coalitional manipulability rates of several voting rules, according to various assumptions about the structure of the population.

Among the studies above, some authors mention the intuition that Condorcet rules have a general trend to be less CM than others [4, 35, 9, 22, 8, 36]. In contrast, some suggest that *Single Transferable Vote* (STV) is one of the least CM among "reasonable" voting rules ever studied [4, 20, 22, 13, 14], despite not being a Condorcet rule.

Recently, Durand et al. [7] gave theoretical insight on this issue. They considered an alteration of any voting rule called its *Condorcification*, by adding the provision that whenever a Condorcet winner exists, she is elected; in other cases, the original rule is used. This idea is a straightforward generalization of Black's method [1], that was proposed to get a Condorcet-consistent version of the *Borda count* rule. Until recently, Condorcification was hardly studied in general, as it was seen as an inelegant way to produce Condorcet methods. However, Durand et al. [7] showed that for a large class of voting rules, their Condorcification is at most as CM as the original rule. In a recent paper, Green-Armytage et al. [15] proved also this result independently. We will recall this result formally in Theorem 1.

Roadmap The rest of this paper is organized as follows.

Section 2 introduces some basic definitions of voting theory and our notations. Section 3 states some previous work about the coalitional manipulability of Condorcet rules, with the purpose of clearing up some possible misinterpretations of these results.

In Section 4, we give the result mentioned above [7]: for a large

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[3] I.e. where at least three candidates are in the image of the voting rule. Some candidates may exist but not be eligible [32].

class of voting rules, their Condorcification is at most as CM as the original rule. We then introduce a new notion, the *resistant Condorcet winner*, characterized by a form of immunity to coalitional manipulation. We use this to show that for a large class of voting rules, their Condorcification is *strictly* less CM than the original rule. Then, we stress on an important consequence of these two results: among a large class of voting rules that share a natural property, when searching for a voting rule with minimal coalitional manipulability, investigations can be restricted to Condorcet rules. In other words, in a class of "reasonable" voting rules, it is possible to have both the Condorcet criterion and a minimal vulnerability to coalitional manipulation.

Section 5 extends our framework by allowing voters to have weak orders or even more general binary relations of preference over the candidates. We show that all previous results still hold true, provided that the notion of Condorcet winner is replaced by what we call *absolute Condorcet winner*, instead of the usual definition.

Finally, in Section 6, we generalize the results to non-ordinal voting rules, especially cardinal ones, among which are Approval voting and Range Voting.

2 Framework

Consider two non-empty finite sets $\mathcal{V}$ and $\mathcal{C}$, whose elements are respectively called *voters* and *candidates*. Some or all of the candidates can also be voters themselves, without impact on our results. Let $V = \text{card}(\mathcal{V})$ and $C = \text{card}(\mathcal{C})$.

$\mathcal{L}$ denotes the set of strict total orders over $\mathcal{C}$, i.e. transitive, irreflexive and complete binary relations. We assume for the moment that each voter v has a strict total order of preference over the candidates, denoted $\mathrm{P}_v \in \mathcal{L}$; this assumption will be relaxed in Section 5. An element P of $\mathcal{L}^\mathcal{V}$ is called a *profile*: for each voter v, it gives her relation of preference P_v. A *voting rule* is a function $f : \mathcal{L}^\mathcal{V} \to \mathcal{C}$ that, to each profile, associates a winning candidate. We say that f is *coalitionally manipulable* (CM) in profile P towards a profile P′ if and only if:

$$\begin{cases} f(\mathrm{P}') \neq f(\mathrm{P}), \\ \forall v \in \mathcal{V}, \left(\mathrm{P}'_v \neq \mathrm{P}_v \Rightarrow f(\mathrm{P}')\ \mathrm{P}_v\ f(\mathrm{P})\right). \end{cases}$$

Denoting $c = f(\mathrm{P}')$, we also say that P is CM *in favor of* c.

$M_f \subseteq \mathcal{L}^\mathcal{V}$ denotes the set of profiles where f is CM. In this paper, our goal is to diminish coalitional manipulability in the sense of inclusion: so, we will say that a voting rule g is at most as CM as f if and only if (iff) $M_g \subseteq M_f$.

Given a profile P and two distinct candidates c and d, we use $\left|c\ \mathrm{P}_v\ d\right|$ as a short notation for the number of voters who prefer c to d. We say that c has a *victory* against d in P, or equivalently that d has a *defeat* against c in P, iff $\left|c\ \mathrm{P}_v\ d\right| > \frac{V}{2}$. We say that a candidate c is *Condorcet winner* in a profile P iff c has a victory against any other candidate in P. We say that P is a *Condorcet profile* iff there is a Condorcet winner in P. We say that a voting rule f meets the *Condorcet criterion* iff for any Condorcet profile P, the elected candidate $f(\mathrm{P})$ is the Condorcet winner; as a language convenience, we also say that f is a *Condorcet rule*.

In this ordinal framework, we now recall some classic voting rules[4]. The definitions below can lead to ties between several candidates; in all the following, we will consider that an arbitrary tie-breaking rule is used.

Positional scoring rules (PSR) Let $w = (w_1, \dots, w_C)$ be a non-increasing and non-constant vector of real numbers. In the PSR of weight vector w, the *score* of a candidate c is defined as $\sum_{v\in\mathcal{V}} w_{r(c,\mathrm{P}_v)}$, where $r(c, \mathrm{P}_v)$ denotes the rank of candidate c in the preference order P_v of voter v. The candidate with highest score is declared the winner. The most studied PSRs are the three following voting rules.

Plurality PSR of weight vector $(1, 0, \dots, 0)$.

Antiplurality PSR of weight vector $(0, \dots, 0, -1)$.

Borda count PSR of weight vector $(C - 1, C - 2, \dots, 0)$.

Two-round system[5] Computing the winner involves two steps or *rounds*. Only the two candidates with highest Plurality scores are selected for the second round, during which each voter grants one point to the candidates she prefers among the two. The candidate with highest score in second round is declared the winner.

Single Transferable Vote (STV) There are $C - 1$ rounds. At each round, the candidate with the lowest Plurality score is eliminated. Plurality scores are updated each time, depending on the eliminated candidates: each voter gives one point to the highest non-eliminated candidate in her order of preference.

Coombs' method As in STV, there are $C-1$ rounds. At each round, the candidate with the lowest Antiplurality score is eliminated.

Bucklin's method The *median rank* of a candidate c is the median of the list $\left(r(c, \mathrm{P}_v)\right)_{v\in\mathcal{V}}$. The candidate with the best (i.e. lowest) median rank is elected. If several candidates have the same median rank, the winner is the candidate to which a highest number of voters assign this rank or better (i.e. lower).

3 Condorcet Rules and Coalitional Manipulability: Facts and Traps

The following classic result relates Condorcet notions and coalitional manipulability: in a Condorcet rule, a Condorcet profile cannot be CM towards another Condorcet profile [23]. But, despite common belief, a Condorcet profile is not necessarily immune to coalitional manipulation, even in a Condorcet rule. Worse, in any Condorcet rule with 3 candidates and 3 voters, there exists at least one CM Condorcet profile. To prove this assertion, consider first the following non-Condorcet profile P′.

$$\mathrm{P}' = \begin{array}{c|c|c} a & b & c \\ b & c & a \\ c & a & b \end{array}$$

We follow the usual convention to represent profiles: for example, the first column above means that voter 1 has the order of preference $a \succ b \succ c$.

If $f(\mathrm{P}') = a$, then consider the following profile P, where only the first voter is different from P′.

$$\mathrm{P} = \begin{array}{c|c|c} a & b & c \\ \mathbf{c} & c & a \\ \mathbf{b} & a & b \end{array}$$

Then candidate c is Condorcet winner in P. But the first voter can manipulate towards profile P′, because she prefers a to c.

[4] For more details, see for example [36].

[5] We consider an instantaneous version of the Two-round system: voters give an order of preference, and the two rounds are computed automatically. In most actual implementations, the voting rule is sligthly different since voters go to the polls once for each round. It is easy to see that the instantaneous version is at most as manipulable (individually or coalitionally) as the version with two actual rounds: in the latter, sincere voting leads to the same outcome, but manipulators have a larger set of available strategies [23].

If $f(\mathrm{P}') = b$ or $f(\mathrm{P}') = c$, we can exhibit a similar example by using the symmetry of profile P'.

This statement still holds true for more than 3 candidates (by adding candidates at the end of all preferences in P' and on top of the first voter's preferences in P). It also extends to 5 voters or more, by replacing the three voters by three groups of voters of approximately equal size[6]. So, in general, it is not true that a Condorcet profile is immune to coalitional manipulation, even in a Condorcet rule.

Another classic result deals with coalitional manipulability in *single-peaked* contexts [23]. We say that a preference order P_v is single-peaked [1] relatively to an order $\mathrm{P}_0 \in \mathcal{L}$ (typically, a left-right political axis) iff for any candidates c, d, e such that $c\ \mathrm{P}_0\ d\ \mathrm{P}_0\ e$, it is impossible to have simultaneously $c\ \mathrm{P}_v\ d$ and $e\ \mathrm{P}_v\ d$. We say that a profile is single-peaked relatively to P_0 iff it is the case for all individual preferences. As made famous by Black [1], in a single-peaked profile with an odd number of voters, there is always a Condorcet winner. Moreover, with an odd number of voters, if a Condorcet rule is restricted to the profiles that are single-peaked relatively to some given order P_0, then the rule is not CM [23].

Despite common belief, this does not mean that in all single-peaked contexts, coalitional manipulation is not an issue, and that Condorcet rules solve the problem. As discussed by Blin and Satterthwaite [2] for Black's rule, for the non-manipulability result to hold, it is important to assume that *sincere preferences* and *ballots* are both *a priori* restricted to be single-peaked relatively to a given order. More recently, Penn et al. [27] considered a framework where profiles are single-peaked, but relatively to an order that is not known *a priori* when designing the voting rule: in particular, each voter is allowed to use any strict total order as her ballot. They show that in that case, for any non-trivial voting rule, at least one single-peaked profile is manipulable (even by a single manipulator).

Thus, in single-peaked contexts, when the order P_0 is not known in advance, it is not *a priori* obvious that Condorcet rules are less prone to coalitional manipulation than the others.

Given these results, it is not clear that Condorcet rules are less CM than others in general. Actually, as mentioned earlier, some studies suggest that STV is generally less CM than most known Condorcet rules [4, 20, 22, 13, 14]. In the following, we will not support this too optimistic idea but a more nuanced one: in a large class of voting rules, it is possible to combine the Condorcet criterion and a minimal vulnerability to coalitional manipulation (even if the first does not necessarily imply the second).

4 Condorcification

We first study Condorcification in the framework of strict total orders, before expanding these results to arbitrary binary relations of preference (Section 5) and to non-ordinal voting rules (Section 6).

4.1 Weak Theorem of Condorcification

We call *Condorcification* of f the voting rule f^* defined as follows.

- If there is a Condorcet winner in profile P, then she is elected by f^*.
- Otherwise, $f^*(\mathrm{P}) = f(\mathrm{P})$.

[6] This example does not extend to 4 voters, because one of the three groups would consist of half the voters. In fact, with $V = 4$ and $C = 3$, it is easy to check that there exists a Condorcet rule where no Condorcet profile is CM: for each non-Condorcet profile, elect an arbitrary candidate who has no defeat (for example, a Plurality winner).

For example, *Black's method* [1] is defined as the Condorcification of the Borda count.

It is easy to check that Condorcification preserves *anonymity* (symmetry of voters), *neutrality* (symmetry of candidates) and *monotonicity* (if a candidate c wins, then if one voter moves c up in her ballot, then c cannot become a loser). For the latter, it is sufficient to remark that in a Condorcet rule, there cannot be a violation of monotonicity involving a Condorcet profile; so, if there exists a non-monotonicity paradox in f^*, it is between two non-Condorcet profiles, so it also exists in f. Of course, f^* meets the Condorcet criterion and all criteria it implies, for example the *majority criterion* (if a candidate is ranked first by a strict majority, then she is elected).

But our main focus in this paper is its effect on coalitional manipulability: Durand et al. [7] and Green-Armytage et al. [15] showed that for an important class of voting rules, their Condorcification is at most as CM as the original rule. To state this result formally, we call a *coalition* a subset of the voters and a *majority coalition* a coalition whose cardinality is strictly greater that $\frac{V}{2}$. We say that a voting rule f meets the *informed majority coalition criterion* (InfMC) iff for any candidate c, for any majority coalition $\mathcal{M}$, for any profile P, there exists a profile P' such that:

$$\begin{cases} \forall v \notin \mathcal{M}, \mathrm{P}'_v = \mathrm{P}_v, \\ f(\mathrm{P}') = c. \end{cases}$$

In other words, any majority coalition may ensure the victory of any candidate, provided they know in advance the other voters' ballots. This criterion appears under different names in several sources: InfMC [7], *Conditional Majority Determination* [15] or without explicit name [3]. It is closely related to Peleg's notion of β-*effectivity* [26].

It is easy to check that most usual voting rules meet InfMC (except some exotic positional scoring rules such as Antiplurality, rarely used in actual settings): Plurality, Two-round system, STV, Borda count, Bucklin's and Coombs' methods, and all Condorcet rules. Among common voting rules, it is interesting to see that most meet InfMC, even those whose usual rationale does not rely on the notion of majority (such as Approval voting, as we will see in Section 6). In practice, this gives a wide scope of application for the following theorem. From a theoretical point of view, we can wonder whether there is a deep reason why most common voting systems meet this criterion; we think that this is an interesting question for future work.

If f meets InfMC, it is easy to prove this property: for any profile P' that is a strong Nash equilibrium for the game defined by f and some profile P, the winner $f(\mathrm{P}')$ has necessarily no defeat in P (i.e., if V is odd, she must be a Condorcet winner)[7]. This gives a first intuition why choosing the Condorcet winner might be a good idea to prevent coalitional manipulation.

The following theorem is mentioned without proof by Durand et al. [7], and Green-Armytage et al. [15] provides a version of the proof that is only valid for strict total orders of preference, as we will discuss in Section 5. We will give a more general proof in Section 6.

Theorem 1 (Weak Condorcification) *If f meets* InfMC*, then its Condorcification is at most as CM as f.*

$$M_{f^*} \subseteq M_f.$$

[7] Actually, the converse is true: this property implies that f meets InfMC [6].

4.2 Strong Theorem of Condorcification

In this section, we give a second Condorcification theorem, stating that for most usual voting rules that do not meet the Condorcet criterion, their Condorcification is not only at most as CM, but *strictly* less CM. In order to prove this, we introduce the notion of *resistant Condorcet winner* (RCW), a candidate that possesses a form of immunity to coalitional manipulation. We say that candidate c is an RCW in profile P iff, for any pair of candidates $d, e \in \mathcal{C} \setminus \{c\}$ (not necessarily distinct from each other):

$$|c \ \mathrm{P}_v \ d \text{ and } c \ \mathrm{P}_v \ e| > \frac{V}{2}.$$

We use this notation: given an assertion $\mathcal{A}(v)$ that depends on voter v, we denote $\left|\mathcal{A}(v)\right| = \mathrm{card}\big(v \in \mathcal{V} \text{ s.t. } \mathcal{A}(v)\big)$.

Proposition 1 (Characterization of the RCW) *Given a profile* P *and a candidate c, the following conditions are equivalent.*

1. *Candidate c is RCW in* P.
2. *For any Condorcet rule f, c is elected by sincere voting, i.e. $f(\mathrm{P}) = c$, and f is not CM in* P.

Proof: $1 \Rightarrow 2$. This part being the easiest, we give only a sketch of proof. Assume that c is RCW in P. Let f be a Condorcet rule. Since c is clearly Condorcet winner in P, we have $f(\mathrm{P}) = c$. Consider a manipulation attempt in favor of a candidate $d \neq c$, i.e. a profile P′ where only voters preferring d to c may change their ballot, whereas those preferring c to d cannot do so. In particular, for any candidate $e \neq c$, voters who simultaneously prefer c to d and c to e in P keep the same ballots in P′; since c is an RCW in P, they guarantee that c still has a victory against e in P′. So, candidate c still appears as a Condorcet winner in P′, she gets elected and the manipulation fails. Hence, f is not CM in P.

Not $1 \Rightarrow$ not 2. Assume that condition 1 is false, i.e. c is not an RCW in P. We can assume however that c is Condorcet winner in P, otherwise it is trivial that condition 2 is false (because we can choose a Condorcet rule f such that $f(\mathrm{P}) \neq c$).

Let (d, e) be a pair of candidates violating the definition of the RCW. Necessarily, $e \neq d$, otherwise c would not be a Condorcet winner. We will exhibit a profile P′ without Condorcet winner and differing from P only by voters preferring d to c. So, it will be possible to choose a Condorcet rule f such that $f(\mathrm{P}') = d$. From this, we will deduce that f is CM in profile P towards P′, in favor of d.

So, let us exhibit such a profile P′. Up to switching roles between d and e, we can assume that e has no victory against d in profile P. Let p be a strict total order of the form: $(d \succ e \succ c \succ$ other candidates). For each voter v preferring d to c in P ("manipulator"), let $\mathrm{P}'_v = p$. For each other voter v ("sincere voter"), let $\mathrm{P}'_v = \mathrm{P}_v$. In the new profile P′, candidate c is not a Condorcet winner because she does not have a victory against e: indeed, the only voters who claim preferring c to e in P′ are those of the sincere voters who already preferred c to e in P, which leads to $\left|c \ \mathrm{P}'_v \ e\right| = \left|c \ \mathrm{P}_v \ d \text{ and } c \ \mathrm{P}_v \ e\right| \leq \frac{V}{2}$. Candidate d is not a Condorcet winner (it is an easy and classic result that her duel against c cannot have been improved by manipulation [23]). Neither can candidate e because she still has no victory against d. And neither can other candidates, because the number of voters who claim preferring c to them has not diminished from P to P′. □

We say that a voting rule meets the *resistant-Condorcet criterion*[8] iff, whenever there is an RCW, she is elected. Clearly, this criterion is weaker than the Condorcet criterion because it constrains the result in a smaller set of profiles.

In practice, all the usual voting rules violating the Condorcet criterion also violate the resistant-Condorcet criterion (for some values of V and C). Indeed, consider a profile P of the following type, with $V = 100$ voters and $C = 17$ candidates.

P =

17	13	14	14	14	14	14
a	c	d_1	d_2	d_4	d_7	d_{11}
	a	c	d_3	d_5	d_8	d_{12}
		a	c	d_6	d_9	d_{13}
			a	c	d_{10}	d_{14}
				a	c	d_{15}
					a	c
Others	Others	Others	Others	Others	Others	Others
c	d_1	d_2	d_4	d_7	d_{11}	a

In the above notation, each column gathers identical voters and its top cell indicates the corresponding number of voters. For each column, the respective positions of candidates denoted "others" is not important for this example. We let the reader check the following. Candidate c is an RCW. However, in any PSR (including Plurality, Borda count and Antiplurality), candidate a has a better score than c. It is also true in Bucklin's method. In the Two-round system, STV or Coombs' method, c is eliminated during the first round. Hence, none of these voting rules meet the resistant-Condorcet criterion.

It is easy to define an artificial example of a voting rule that meets the resistant-Condorcet criterion but not the Condorcet criterion: for example, consider a rule electing the RCW when she exists, and a constant candidate otherwise. But the observation above tends to show that a voting rule that was not designed to elect all Condorcet winners has no "natural" reason to elect the resistant ones.

Now, we have the necessary tools to state and prove the strong theorem of Condorcification.

Theorem 2 (Strong Condorcification) *If f meets* InfMC *but not the resistant-Condorcet criterion, then its Condorcification f^* is strictly less CM than f:*

$$M_{f^*} \subsetneq M_f.$$

Proof: The weak theorem of Condorcification (Th. 1) ensures the inclusion. Since f does not meet the resistant-Condorcet criterion, there exists a profile P, a candidate c who is RCW in P, such that $f(\mathrm{P}) \neq c$. Since c is a Condorcet winner, a strict majority of voters prefer c to $f(\mathrm{P})$; by InfMC, it implies that f is CM in P in favor of c. In contrast, Proposition 1 ensures that f^* is not CM in P. Hence, the inclusion is strict. □

In particular, Theorem 2 proves that for Plurality, Two-round system, STV, Borda count, Bucklin's and Coombs' methods, their Condorcification is strictly less CM than the original rule.

The reader may have noticed that the implication $2 \Rightarrow 1$ in Proposition 1 is not necessary to prove Theorem 2. We mentioned it to show the deep connection between the property of being an RCW and the immunity to coalitional manipulation in the Condorcet rules.

4.3 Optimality Corollary

Up to now, we have considered a given voting rule f and compared the set of CM profiles for f and for its Condorcification f^*. At first look, these results may suggest to use voting rules such as the Condorcification of Plurality, STV, etc. However, we think that it is not the main consequence of the Condorcification theorems. Indeed, they imply the following corollary. As a notational convenience, the set of voting rules meeting InfMC is also denoted by InfMC.

[8] We use a dash to stress on the fact that the adjective *resistant* applies to the word Condorcet, not criterion.

Corollary 1 (Optimality) *Let us consider the function:*

$$M : \left| \begin{array}{lcl} \text{InfMC} & \to & \mathcal{P}(\mathcal{L}^V) \\ f & \to & M_f \end{array} \right.$$

returning, for each voting rule f meeting InfMC*, the set M_f of its CM profiles.*

Let $A \in \mathcal{P}(\mathcal{L}^V)$ be a minimal value of M, i.e. a set of profiles such that at least one voting rule $f \in$ InfMC *meets $M_f = A$, but no rule $f \in$* InfMC *meets $M_f \subsetneq A$. Then:*

- *Any rule $f \in$* InfMC *meeting $M_f = A$ meets the resistant-Condorcet criterion.*
- *There exists a Condorcet rule f such that $M_f = A$.*

In order to understand the scope of this theorem, let us notice that the function M may have several minima that are not comparable, because the inclusion relation over $\mathcal{P}(\mathcal{L}^V)$ is not a total order. In other words, there may be different rules f and g such that no voting rule is less CM than f or g, but whose sets of CM profiles, M_f and M_g respectively, are not comparable.

This corollary can be summed up this way: when looking for a voting rule meeting InfMC with minimal coalitional manipulability, then investigations *must* be restricted to rules meeting the resistant-Condorcet criterion and *can* be restricted to Condorcet rules. In other words, this corollary answers the main question of this paper: when restricting to "reasonable" voting rules, in the sense that they meet InfMC, it is possible to have both the Condorcet criterion and a minimal vulnerability to coalitional manipulation.

5 Arbitrary Binary Relations

Now, let $\mathcal{P}$ be a subset of the binary relations over the candidates. $\mathcal{P}$ will represent the set of relations we assume possible for each voter. The relation $\mathrm{P}_v \in \mathcal{P}$ of a voter v is interpreted in the following way: for any pair of distinct candidates (c, d), the assertion $c \ \mathrm{P}_v \ d$ means that when d is the winner of sincere voting, v may be interested in taking part in a coalitional manipulation in favor of c.

In most usual models, this relation is identified with the voter's binary relation of strict preference over the candidates. With this interpretation, it is natural to assume that it is antisymmetric: v cannot strictly prefer c to d and d to c in the same time. However, with the general interpretation of P_v as an inclination to manipulate, it is conceivable to have a "crazy manipulator" who wants to manipulate for c when d would win by sincere voting, and vice-versa. Moreover, the antisymmetry assumption is not needed for the proofs of our results. So, for the sake of generality, we will not make this assumption in the rest of this paper. That being said, should the reader be confused with the absence of antisymmetry assumption, she can read all the following with this additional assumption in mind and the usual interpretation of P_v as a strict preference.

Since it is common to identify the inclination to manipulate with strict preferences, we will use the following language shortcut: when $c \ \mathrm{P}_v \ d$, we will go on saying that voter v *prefers* c to d.

Typically, $\mathcal{P}$ can be the set of strict total orders like in previous sections, or the set of strict weak orders (negatively transitive, irreflexive and antisymmetric relations), or the set of preferences that are single-peaked relatively to a given order, etc. But in the general case, absolutely no assumption is made about $\mathcal{P}$. A relation $\mathrm{P}_v \in \mathcal{P}$ may not be complete (e.g. strict weak orders). It may not be transitive either: voter v may prefer candidate a to b, b to c and c to a.

In this first extension of the framework, a *voting rule* is a function $f : \mathcal{P}^V \to \mathcal{C}$. In this case, there are at least two natural generalizations of the Condorcet winner.

1. We say that a candidate c is an *absolute Condorcet winner* iff for any other candidate d, she has an *absolute victory* against d, in the sense that $\left| c \ \mathrm{P}_v \ d \right| > \frac{V}{2}$ and $\left| d \ \mathrm{P}_v \ c \right| \leq \frac{V}{2}$. The main motivation for the second condition is to ensure the uniqueness of the absolute Condorcet winner in the unusual models where non-antisymmetric relations are allowed. For antisymmetric relations, it can safely be omitted, because it becomes redundant with the first condition.
2. We say that candidate c is a *relative Condorcet winner* iff for any other candidate d, she has a *relative victory* against d, in the sense that $\left| c \ \mathrm{P}_v \ d \right| > \left| d \ \mathrm{P}_v \ c \right|$.

When preferences are strict total orders, these two notions are obviously equivalent, and both amount to the notion of the Condorcet winner that we have used up to now.

Similarly, there are also two natural notions that generalize Condorcification: *absolute Condorcification* and *relative Condorcification*, which respectively add a preliminary test about the existence of an absolute or a relative Condorcet winner and elect her if she exists.

In Section 6, we will prove that the weak theorem of Condorcification (Th. 1) still holds in the general case when considering the absolute Condorcification. For this reason, in the following, we will use the terms *Condorcet winner* and *Condorcification* for the absolute version of these notions.

In contrast, we will now show that the weak theorem of Condorcification (Th. 1) is not true when replacing f^* by the relative Condorcification of f, denoted by f^{rel}. In other words, some profiles may be CM in f^{rel} whereas they are not CM in f.

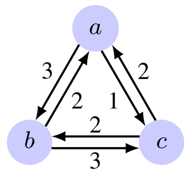
Let us start with a voting rule that is a bit artificial but makes it possible to prove this concisely. Assume that preferences are strict weak orders. Let f be the voting rule that we call *Condorcet-dean*:

1. If there is an absolute Condorcet winner, then she is elected.
2. Otherwise, a constant candidate called the *dean* (say, candidate a) is elected.

Obviously, this rule meets InfMC.

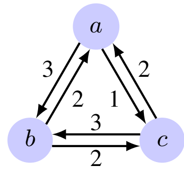
Consider $V = 5$, $C = 3$ and the following profile P.

$$\mathrm{P} = \begin{array}{c|c|c|c|c} a & a, c & b & b & c \\ b & b & a, c & c & a \\ c & & & a & b \end{array}$$



The above notation on the left means for example that the second voter is indifferent between candidates a and c, which she prefers to b. On the right is the *weighted majority graph*: nodes are the candidates, and for each pair of distinct candidates (x, y), there is a directed edge from x to y whose weight is $\left| x \ \mathrm{P}_v \ y \right|$. It is easy to check that $f(\mathrm{P}) = f^{\text{rel}}(\mathrm{P}) = a$ and that f is not CM in P. Now, consider the following profile P′, where only the fourth voter changes her ballot.

$$\mathrm{P}' = \begin{array}{c|c|c|c|c} a & a, c & b & \mathbf{c} & c \\ b & b & a, c & \mathbf{b} & a \\ c & & & a & b \end{array}$$



Since candidate c is the relative Condorcet winner, we have $f^{\text{rel}}(\text{P}') = c$. In conclusion of this example, f^{rel} is CM in P towards P′ in favor of candidate c (it is even *individually* manipulable), whereas f is not CM in P. Hence, Theorem 1 does not generalize when considering relative Condorcification.

While the voting rule used above is exotic, we can produce a similar counter-example with f being the Single Transferable Vote (STV), with weak orders allowed. If a voter has two candidates or more tied on top of her ballot, her vote is equally shared between these candidates. Let us consider the following profile P with $V = 100$ voters.

P =

12	11	25	12	12	28
a	a	b	b	c	d
b	d	a, c	c	a	a, c
c	c	d	a	d	b
d	b		d	b	

In f, candidates c, then d, then b are successively eliminated hence $f(\text{P}) = a$. First, let us show that f is not manipulable in P.

- In favor of candidate b: even if she reached the last round, she would lose against any other candidate (by 63 or 51 votes).
- In favor of candidate c: the fourth and fifth groups of voters are interested ($12 + 12$ voters). For candidate c not being eliminated during the first round, it is necessary that 23 or 24 manipulators put her on top of her ballot and that candidate a is eliminated (it is not possible to eliminate candidate b or d). In the second round, since candidates b and d have more that one third of the votes each (37 and 39, respectively), candidate c is eliminated.
- In favor of candidate d: only the sixth group (28 voters) is interested. During the first round, they cannot simultaneously save candidates c and d from elimination: indeed, since candidate a receives 23 votes, manipulators and voters sincerely casting a ballot for candidates c or d would need to have at least $2 \times 23 = 46$ votes, but they have only $12 + 28 = 40$. Hence, since candidate d must stay, candidate c must be eliminated in the first round. In the second round, candidates a and b have more than one third of the votes each (35 and 37 respectively), so candidate d is eliminated.

Since candidate c is the relative Condorcet winner, we have $f^{\text{rel}}(\text{P}) = c$. Let us consider the following profile P′, an attempt of manipulation in favor of candidate a.

P′ =

12	11	25	12	12	28
a	a	b	b	c	d
b	d	a, c	c	a	a, c
$\mathbf{d}$	$\mathbf{b}$	d	a	d	b
$\mathbf{c}$	$\mathbf{c}$		d	b	

For the point of view of STV, the counting resolve the same way and we have $f(\text{P}') = a$. And since there is no relative Condorcet winner, we also have $f^{\text{rel}}(\text{P}') = a$. In conclusion, f^{rel} is CM in P towards P′ in favor of a, whereas f is not CM in P.

Remark that the two examples are not constructed in the same way. In Condorcet-dean, f and f^{rel} give the same output in the initial profile but different outcomes in the manipulated profile. In constrast, in the example of STV, f and f^{rel} return different outputs in the initial profile but the same output in the manipulated profile. Both types of problems can occur with the relative Condorcification.

We insist again on the importance of distinguishing the notions of relative and absolute Condorcet winner when dealing with more general preferences than strict total orders. For example, the proof of the weak Condorcification Theorem presented by Green-Armytage et al. [15] relies on relative Condorcification, so it cannot be adapted to weak orders. In details, at the end of point 2 of their proof, it is established that no candidate B is preferred by a strict majority to some candidate A. In point 3 of the proof, it is deduced from this that no candidate B can be a (relative) Condorcet winner. This implication fails with weak orders: indeed, a candidate B can be a relative Condorcet winner and be preferred to A by only 45% of the voters, whereas A is preferred to B by 40% of the voters.

6 Generalization

Up to now, we considered only *ordinal* voting rules: we use this term in an extended sense, meaning that the winner depends only on the binary relations of preference (whether they are orders or not, strictly speaking). In this section, we generalize the previous results to non-ordinal voting rules, where the ballot of a voter may contain information that is not included in her order of preference, for example *Range Voting* (where each voter assigns a grade to each candidate in a set of authorized values, and the candidate with highest average grade is elected).

An *electoral space* is defined by:

- Two non-empty finite sets $\mathcal{V}$ and $\mathcal{C}$;
- For each voter $v \in \mathcal{V}$, a non-empty set Ω_v of her possible *states*;
- For each voter $v \in \mathcal{V}$, a function $\text{P}_v : \Omega_v \to \mathcal{R}$, where $\mathcal{R}$ denote the set of binary relations over the candidates.

Denote $\Omega = \prod_{v \in \mathcal{V}} \Omega_v$. An element $\omega \in \Omega$ is called a *configuration*: for each voter v, it gives her state ω_v. Such an electoral space is denoted by $(\mathcal{V}, \mathcal{C}, \Omega, \text{P})$, or just Ω in short. A *voting rule* (over an electoral space Ω) is a function $f : \Omega \to \mathcal{C}$.

As an example, for Range Voting, we can consider the following model: for each voter v, her state ω_v is a vector of grades, one for each candidate. Her order of preference $\text{P}_v(\omega_v)$ is the one induced by ω_v, in the sense that she prefers a candidate c to a candidate d iff she assigns a strictly higher grade to c than d. This model is especially relevant if there is a great number of authorized grades: in that case, it is reasonable to consider that if a voter sincerely assigns the same grade to two candidates, then she is indifferent between them.

But this assumption is not reasonable when there is a small number of authorized grades (the extreme case being Approval Voting, which can be seen as Range Voting with only grades 0 and 1). In any case, the following model can also be considered. For each voter v, her state ω_v is a pair (p_v, g_v), where p_v is a strict weak order of preference over the candidates and g_v is a vector of C grades that is coherent with p_v, in the sense that for any two candidates c and d, if $c \ p_v \ d$, then $g_v(c) \geq g_v(d)$. The function P_v is then defined by $\text{P}_v(p_v, g_v) = p_v$.

The framework of electoral spaces is a generalization of the ordinal framework. Indeed, consider the model of Section 5, where $\mathcal{P}$ is the set of binary relations that are possible for any voter. This can be modeled by an electoral space where for each voter v, $\Omega_v = \mathcal{P}$ and P_v is the identity function.

We say that f is *coalitionally manipulable* (CM) in configuration ω towards a configuration ψ iff:

$$\begin{cases} f(\psi) \neq f(\omega), \\ \forall v \in \mathcal{V}, \big(\psi_v \neq \omega_v \Rightarrow f(\psi) \ \text{P}_v(\omega_v) \ f(\omega)\big). \end{cases}$$

The notions of InfMC, Condorcet winner and Condorcification extend easily to this new framework. We denote by $M_f \subseteq \Omega$ the set of configurations where f is CM.

6.1 Weak Theorem of Condorcification

Theorem 3 (Weak Condorcification) *If f meets* InfMC*, then its Condorcification is at most as CM as f.*

$$M_{f^*} \subseteq M_f.$$

Remark that if ψ is a Condorcet configuration, then changing its result to the Condorcet winner cannot worsen manipulability in ψ (i.e. make it manipulable if it was not in the original rule). Indeed, if $f(\psi)$ is not the Condorcet winner, then f is CM in ψ anyway, because f meets InfMC; so, the modified voting rule cannot do worse. However, this simple remark is not sufficient to prove the theorem: it does not exclude the possibility that changing the result in ψ make another configuration ω manipulable towards ψ.

Proof: Suppose that f^* is CM in a configuration ω towards a configuration ψ, but f is not CM in ω.

Let $c = f(\omega)$. For any $d \in \mathcal{C} \setminus \{c\}$, we have $\left|d \ \mathrm{P}_v(\omega_v) \ c\right| \leq \frac{V}{2}$: otherwise, since f meets InfMC, f would be CM in ω in favor of d.

As a consequence, no other candidate than c is an absolute Condorcet winner in ω. By definition of the Condorcification f^*, this leads to $f^*(\omega) = c$.

Now, let $d = f^*(\psi)$. We already know that $\left|d \ \mathrm{P}_v(\omega_v) \ c\right| \leq \frac{V}{2}$. Voters who do not prefer d to c do not modify their ballots from ω to ψ, hence $\left|d \ \mathrm{P}_v(\psi_v) \ c\right| \leq \left|d \ \mathrm{P}_v(\omega_v) \ c\right| \leq \frac{V}{2}$. As a consequence, d is not an absolute Condorcet winner in ψ. So, by definition of the Condorcification f^*, there is no absolute Condorcet winner in ψ and we have $f(\psi) = d$.

Hence, $f(\omega) = f^*(\omega)$ and $f(\psi) = f^*(\psi)$ so f is CM in ω towards ψ: this is a contradiction. □

6.2 Strong Theorem of Condorcification

For the strong theorem of Condorcification (Th. 2), the key point is to generalize correctly the central notion of RCW. In the most general case, we say that candidate c is an RCW in configuration ω iff, for any pair of candidates $d, e \in \mathcal{C} \setminus \{c\}$ (not necessarily distinct from each other):

$$\begin{cases} \left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } c \ \mathrm{P}_v(\omega_v) \ e\right| > \dfrac{V}{2}, & (1) \\ \left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } \mathrm{not}\big(e \ \mathrm{P}_v(\omega_v) \ c\big)\right| \geq \dfrac{V}{2}, & (2) \end{cases}$$

With the (usual) assumption that preferences are antisymmetric, Eq. (2) becomes redundant and the definition amounts only to:

$$\left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } c \ \mathrm{P}_v(\omega_v) \ e\right| > \frac{V}{2}.$$

Proposition 1, characterizing the RCW, generalizes as follows.

Proposition 2 (Characterization of the RCW) *Given a configuration ω and a candidate c, consider the following conditions.*

1. *Candidate c is RCW in ω.*
2. *For any Condorcet rule f, c is elected by sincere voting, i.e. $f(\omega) = c$, and f is not CM in ω.*

We have: 1 $\Rightarrow$ 2. If all strict total orders are authorized for any voter, i.e. if $\forall v \in \mathcal{V}, \mathcal{L} \subseteq \mathrm{P}_v(\Omega_v)$, then the converse 2 $\Rightarrow$ 1 is true.

This theorem states that the converse implication 2 $\Rightarrow$ 1 is true, for example, if for each voter v, her set $\mathrm{P}_v(\Omega_v)$ of possible binary relations of preferences is the set of strict weak orders, since it includes the set of strict total orders.

In order to have the converse implication 2 $\Rightarrow$ 1, it is not possible to omit, in condition 2, the assumption that c is elected in any Condorcet rule (or, equivalently, that c is a Condorcet winner). Otherwise, one may consider a configuration ω where all voters are indifferent between all candidates, i.e. all their binary relations of preference are empty. In that case, obviously, no voting rule is manipulable in ω, but no candidate is RCW.

Proof: 1 $\Rightarrow$ 2. The proof is essentially the same as in proposition 1.

Not 1 $\Rightarrow$ not 2. Assume that condition 1 is false, i.e. c is not an RCW. As in the proof of proposition 1, we can assume however that c is a Condorcet winner, otherwise it is trivial that condition 2 is false. We will prove that there exists a Condorcet rule f that is CM in ω.

Since c is not RCW, at least one of equations (1) or (2) from the definition is not met. We distinguish three cases: **A.** Eq. (2) is not met; **B.** Eq. (1) is not met for some $e = d$; or **C.** Eq. (1) is not met with $e \neq d$.

In each case, the principle is the same as in the proof of proposition 1: exhibit a configuration ψ with no Condorcet winner, differing from ω only for some voters who prefer d to c. As a consequence, it is possible to choose a Condorcet rule f such that $f(\psi) = d$. Finally, f is CM in ω towards ψ in favor of d.

Case A. If there exists some candidates d and e such that Eq. (2) is not met, it means that $\left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } \mathrm{not}\big(e \ \mathrm{P}_v(\omega_v) \ c\big)\right| < \frac{V}{2}$. Remark that $e \neq d$, otherwise we would have $\left|d \ \mathrm{P}_v(\omega_v) \ c\right| > \frac{V}{2}$, implying that c is not Condorcet winner. Up to switching roles between d and e, we can assume that e does not have an absolute victory against d in ω. Let p be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates). For each voter v preferring d to c in ω ("manipulator"), we can choose ψ_v such that $\mathrm{P}_v(\psi_v)= p$, thanks to the assumption that all strict total orders are authorized. For each other voter v ("sincere voter"), let $\psi_v = \omega_v$. In the new configuration ψ, candidate c is not a Condorcet winner, because she is defeated by e: indeed, the only voters that claim not preferring e to c in ψ are those of the sincere voters who already did so in ω; formally, $\left|\mathrm{not}(e \ \mathrm{P}_v(\psi_v) \ c)\right| = \left|\mathrm{not}(d \ \mathrm{P}_v(\omega_v) \ c) \text{ and } \mathrm{not}(e \ \mathrm{P}_v(\omega_v) \ c)\right| < \frac{V}{2}$, which translates to $\left|e \ \mathrm{P}_v(\psi_v) \ c\right| > \frac{V}{2}$. Candidate d cannot appear as a Condorcet winner (because her duel against c cannot have been improved by manipulation [23]). Neither can candidate e because she still has no absolute victory against d. And neither can other candidates, because the number of voters who claim preferring c to them has not decreased.

Case B. If Eq. (1) is not met for some $e = d$, it means that $\left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } c \ \mathrm{P}_v(\omega_v) \ d\right| \leq \frac{V}{2}$. Let p be a strict total order of the form: ($d \succ c \succ$ other candidates). For each voter v preferring d to c in ω ("manipulator"), we can choose ψ_v such that $\mathrm{P}_v(\psi_v)= p$, thanks to the assumption that all strict total orders are authorized. For each other voter v ("sincere voter"), let $\psi_v = \omega_v$. In the new configuration ψ, candidate c is not a Condorcet winner, because she does not have a victory against d: indeed, the only voters that claim preferring c to d in ψ are those of the sincere voters who already did so in ω; formally, $\left|c \ \mathrm{P}_v(\psi_v) \ d\right| = \left|\mathrm{not}\big(d \ \mathrm{P}_v(\omega_v) \ c\big) \text{ and } c \ \mathrm{P}_v(\omega_v) \ d\right| \leq \frac{V}{2}$. Candidate d cannot appear as a Condorcet winner (because her duel against c cannot have been improved by manipulation [23]). And neither can other candidates, because the number of voters who claim preferring c to them has not decreased.

Case C. Remains the case where Eq. (1) is not met, with $e \neq d$. For any real number X, we will denote by $\lfloor X \rfloor$ (resp. $\lceil X \rceil$) the floor (resp. ceiling) function applied to X.

For any pair of candidates x and y, we will write:

- $x \; \mathrm{I}_v(\omega_v) \; y$ iff not $x \; \mathrm{P}_v(\omega_v) \; y$ and not $y \; \mathrm{P}_v(\omega_v) \; x$ (indifference).
- $x \; \mathrm{PP}_v(\omega_v) \; y$ iff $x \; \mathrm{P}_v(\omega_v) \; y$ and not $y \; \mathrm{P}_v(\omega_v) \; x$ (antisymmetric part of preferences: this is equivalent to $x \; \mathrm{P}_v \; y$ when the usual assumption is made that preferences are antisymmetric).
- $x \; \mathrm{MP}_v(\omega_v) \; y$ iff $x \; \mathrm{P}_v(\omega_v) \; y$ and $y \; \mathrm{P}_v(\omega_v) \; x$ (mutual preference: this cannot happen when the usual assumption is made that preferences are antisymmetric).

As a notational convenience, we will omit the configuration when it is ω (and not when it is ψ): for example, $x \; \mathrm{P}_v \; y$ means $x \; \mathrm{P}_v(\omega_v) \; y$.

We denote $A_{cd}(\omega) = \left| c \; \mathrm{P}_v(\omega_v) \; d \right|$: it is the number of voter who prefer c to d in ω.

In this third case, Eq. (1) is not met, with $e \neq d$. Denoting $B = \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{P}_v \; e \right|$, it means that $B \leq \frac{V}{2}$. Using case A, we can assume, however, that Eq. (2) is met.

We will see that in the final configuration ψ, we can ensure that there is a victory neither for c against e, nor for e against c.

Let p be a strict total order of the form: ($d \succ e \succ c \succ$ other candidates).

Let p' be a strict total order of the form: ($d \succ c \succ e \succ$ other candidates).

Since c is Condorcet winner, we have $A_{ce}(\omega) > \frac{V}{2}$, so:

$$\left| d \; \mathrm{P}_v \; c \text{ and } c \; \mathrm{P}_v \; e \right| > \frac{V}{2} - B \geq 0.$$

As a consequence, we can choose $\lfloor \frac{V}{2} \rfloor - B$ voters among the manipulators (voters preferring d to c in ω); for each of them, denoted v, choose ψ_v such that $\mathrm{P}_v(\psi_v) = p'$. For each other manipulator v, choose ψ_v such that $\mathrm{P}_v(\psi_v) = p$. Finally, for each voter who prefers c to d in ω ("sincere voter"), let $\psi_v = \omega_v$.

Then, we have:

$$A_{ce}(\psi) = B + \left(\left\lfloor \frac{V}{2} \right\rfloor - B \right) = \left\lfloor \frac{V}{2} \right\rfloor, \tag{3}$$

so c has no victory against e.

By the way, Eq. (1) is not met for this pair (d, e) but Eq. (2) is met, which respectively translate to the first and second following equations:

$$\begin{cases} \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{PP}_v \; e \right| + \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{MP}_v \; e \right| \leq \left\lfloor \frac{V}{2} \right\rfloor, \\ \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{PP}_v \; e \right| + \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{I}_v \; e \right| \quad \geq \left\lceil \frac{V}{2} \right\rceil, \end{cases}$$

hence, by subtraction:

$$\left| \mathrm{not}(d \mathrm{P}_v \, c) \text{ and } c \mathrm{MP}_v \, e \right| - \left| \mathrm{not}(d \mathrm{P}_v \, c) \text{ and } c \mathrm{I}_v \, e \right| \leq \left\lfloor \frac{V}{2} \right\rfloor - \left\lceil \frac{V}{2} \right\rceil.$$

Thanks to our assumptions on the manipulators' ballots in ψ, the only voters who claim preferring mutually c to e or be indifferent between these two candidates in ψ are those of the sincere voters who did so in ω. Formally:

$$\begin{cases} \left| c \; \mathrm{MP}_v(\psi_v) \; e \right| = \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{MP}_v \; e \right|, \\ \left| c \; \mathrm{I}_v(\psi_v) \; e \right| \quad = \left| \mathrm{not}(d \; \mathrm{P}_v \; c) \text{ and } c \; \mathrm{I}_v \; e \right|. \end{cases}$$

By substitution in the previous equation, this leads to:

$$\left| c \; \mathrm{MP}_v(\psi_v) \; e \right| - \left| c \; \mathrm{I}_v(\psi_v) \; e \right| \leq \left\lfloor \frac{V}{2} \right\rfloor - \left\lceil \frac{V}{2} \right\rceil. \tag{4}$$

As a general remark, it is easy to prove that:

$$A_{ec}(\psi) + A_{ce}(\psi) = V + \left| c \; \mathrm{MP}_v(\psi_v) \; e \right| - \left| c \; \mathrm{I}_v(\psi_v) \; e \right|. \tag{5}$$

Substituting equations (3) and (4) in equation (5), we deduce:

$$A_{ec}(\psi) \leq V + \left\lfloor \frac{V}{2} \right\rfloor - \left\lceil \frac{V}{2} \right\rceil - \left\lfloor \frac{V}{2} \right\rfloor = \left\lfloor \frac{V}{2} \right\rfloor,$$

so e has no victory against c.

To sum up, neither c nor e can be Condorcet winner. For the same reasons as in previous cases, neither can d nor any other candidate. □

As a corollary, the strong theorem of Condorcification (Th. 2) still holds true in the general case (remind that f^* designates the absolute Condorcification). This also implies the optimality corollary (Cor. 1) in this more general framework.

Consequently, even in a broader framework when non-ordinal voting are authorized, our main message still holds. In the class InfMC, when searching for a voting rule with minimal coalitional manipulability, investigations can be restricted to Condorcet rules. In other words, it is possible to have both the Condorcet criterion and a minimal vulnerability to coalitional manipulation.

7 Conclusion

We recalled the weak theorem of Condorcification, initially stated by Durand et al. [7] and Grenn-Armytage et al. [15]: for all voting rules that meet the informed majority coalition criterion, their Condorcification is at most as CM as the original rule (Th. 1). Then we introduced the notion of resistant Condorcet winner and we used it to prove the strong theorem of Condorcification (Th. 2): for a large class of voting systems, the improvement provided by Condorcification is strict. We think that the most important consequence of these results is the optimality corollary (Cor. 1): when searching for a "reasonable" voting rule (i.e. meeting InfMC) with minimal manipulability, investigations *must* be restricted to voting rules meeting the resistant-Condorcet criterion and *can* be restricted to Condorcet rules.

When preferences are not limited to strict total orders, and in particular when they are strict weak orders, we showed that all previous results hold, provided that the notions of Condorcet winner and resistant Condorcet winner are generalized adequately. In particular, we showed that the weak theorem of Condorcification (and, as a consequence, the strong theorem) becomes false when considering the usual notion of relative Condorcet winner, but holds true when using the absolute Condorcet winner.

Finally, we showed that all our results extend to non-ordinal voting rules, and in particular cardinal voting rules such as Approval voting and Range voting. In particular, we presented a new proof of the weak theorem of Condorcification (Th. 3) that covers this most general model.

For future work, it would be interesting to evaluate quantitatively the difference of manipulability between a voting rule and its Condorcification: that could be done using a theoretical approach or computer simulations.

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