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Belief Contraction Within Fragments of Propositional Logic

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Abstract. Recently, belief change within the framework of fragments of propositional logic has gained attention. In the context of revision it has been proposed to refine existing operators so that they operate within propositional fragments, and that the result of revision remains in the fragment under consideration. In this paper we generalize this notion of refinement to belief change operators. Whereas the notion of refinement allowed one to define concrete rational operators adapted to propositional fragments in the context of revision and update, it has to be specified for contraction. We propose a specific notion of refinement for contraction operators, called *reasonable refinement*. This allows us to provide refined contraction operators that satisfy the basic postulates for contraction. We study the logical properties of reasonable refinements of two well-known model-based contraction operators. Our approach is not limited to the Horn fragment but applicable to many fragments of propositional logic, like Horn, Krom and affine fragments.

keywords: Belief change, belief contraction, fragments of propositional logic, knowledge representation and reasoning.

1 INTRODUCTION

Belief change in knowledge representation for artificial intelligence studies how a rational agent may modify his beliefs in presence of new information. Belief contraction is a belief change operation that occurs when some beliefs are retracted but no new information is added. Within the symbolic frameworks, where an agent's beliefs are represented by theories, the AGM paradigm [1, 12] became a standard that provides rational postulates any reasonable belief change operator, in particular any contraction operator, should satisfy. When a theory is represented by a propositional formula, Katsuno and Mendelzon [18] reformulated some of the AGM postulates. More recently Caridroit et al. [3] provided a complete reformulation of the AGM postulates for contraction and proposed a representation theorem that characterizes contraction operations in terms of total preorders over interpretations.

Belief contraction has been studied within the framework of propositional logic and several concrete belief contraction operators have been proposed [1, 12, 13, 11, 21, 14]. More recently, belief contraction has been investigated in the Horn fragment and several families of concrete contraction operators have been proposed [9, 24, 2, 26, 8, 27]. Our goal is to provide new contraction operators that operate in various fragments of propositional logic (including, but not restricted to, the Horn fragment).

The motivation of such a study is twofold. First, in many applications, the language is restricted *a priori*. For instance, a rule-based formalization of expert knowledge is much easier to handle for standard users. Second, some fragments of propositional logic allow for efficient reasoning methods, and then an outcome of contraction within such a fragment can be evaluated efficiently.

We generalize the notion of refinement, initially defined for revision [6], to any belief change operator defined from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ where $\mathcal{L}$ denotes propositional logic. A refinement adapts a belief change operator defined in a propositional setting such that it can be applicable in a propositional fragment. The basic properties of a refinement are first to guarantee the outcome of the belief change operation to remain within the fragment and second to approximate the behavior of the original belief change operator, in particular to keep the behavior of the original operator unchanged if the result already fits in the fragment. We characterize these refined operators in a constructive way.

We study the notion of refinement for contraction operators. Contrary to the case of revision and update [6, 5], the refined contraction operators do not necessarily satisfy the basic postulates for contraction. In order to overcome this problem, we introduce a specific notion of refinement for contraction operators, called *reasonable refinement*. This specification allows us to provide concrete rational contraction operators obtained from known model-based contraction operators. We focus on the reasonable refinements of contraction operators defined from Dalal's and Satoh's revision operators within the Horn, Krom and affine fragments. We study the logical properties of these operators in terms of satisfaction of postulates for contraction.

An important contribution of our study is that it provides new rational belief contraction operators that work within propositional fragments. In the Horn case, they do not coincide with any contraction operator previously proposed in the literature.

2 PRELIMINARIES

2.1 Propositional logic

Let $\mathcal{L}$ be the language of propositional logic built on an infinite countable set of variables (atoms) and equipped with standard connectives $\rightarrow$, $\oplus$, $\vee$, $\wedge$, $\neg$, and constants $\top$, $\bot$. A literal is an atom or its negation. A clause is a disjunction of literals. A clause is called *Horn* if at most one of its literals is positive; *Krom* if it consists of at most two literals. An $\oplus$-clause is defined like a clause but using exclusive- instead of standard-disjunction. We identify the following subsets of $\mathcal{L}$: $\mathcal{L}_{Horn}$ as the set of all formulas in $\mathcal{L}$ being conjunctions of Horn clauses; $\mathcal{L}_{Krom}$ as the set of all formulas in $\mathcal{L}$ being conjunctions of Krom clauses; and $\mathcal{L}_{\mathit{Affine}}$ as the set of all formulas in $\mathcal{L}$ being conjunctions of $\oplus$-clauses. In what follows we sometimes just talk about arbitrary fragments $\mathcal{L}' \subseteq \mathcal{L}$.

Let $\mathcal{U}$ be a finite set of atoms. An interpretation is represented either by a set $m \subseteq \mathcal{U}$ of atoms (corresponding to the variables set to true) or by its corresponding characteristic bit-vector of length $|\mathcal{U}|$. For instance if we consider $\mathcal{U} = \{x_1, \ldots, x_6\}$, the interpretation

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$x_1 = x_3 = x_6 = 1$ and $x_2 = x_4 = x_5 = 0$ will be represented either by $\{x_1, x_3, x_6\}$ or by $(1, 0, 1, 0, 0, 1)$.

For any formula ϕ, let $\mathrm{Var}(\phi)$ denote the set of variables occurring in ϕ. As usual, if an interpretation m defined over $\mathcal{U}$ satisfies a formula ϕ such that $\mathrm{Var}(\phi) \subseteq \mathcal{U}$, we call m a model of ϕ. By $\mathrm{Mod}(\phi)$ we denote the set of all models (over $\mathcal{U}$) of ϕ. Moreover, $\psi \models \phi$ if $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\phi)$ and $\psi \equiv \phi$ (ϕ and ψ are equivalent) if $\mathrm{Mod}(\psi) = \mathrm{Mod}(\phi)$. For fragments $\mathcal{L}' \subseteq \mathcal{L}$, we also use $T_{\mathcal{L}'}(\psi) = \{\phi \in \mathcal{L}' \mid \psi \models \phi\}$.

2.2 Characterizable fragments of propositional logic

Let us define the fragments of propositional logic we are interested in. This requires some formal definition.

Let $\mathcal{B}$ be the set of Boolean functions $\beta\colon \{0,1\}^k \to \{0,1\}$ with $k \geq 1$, that have the following properties:

- *symmetry*, i.e., for all permutation σ, $\beta(x_1, \ldots, x_k) = \beta(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$, and
- 0- and 1-*reproduction*, i.e. for every $x \in \{0,1\}$, $\beta(x, \ldots, x) = x$.

Examples of such functions are: the binary AND function denoted by $\wedge$; the binary OR function denoted by $\vee$; the ternary MAJORITY function, $\mathrm{maj}_3(x, y, z) = 1$ if at least two of the variables x, y, and z are set to 1, and 0 otherwise; and the ternary XOR function $\oplus_3(x, y, z) = x \oplus y \oplus z$.

Recall that we consider interpretations also as bit-vectors. We thus extend Boolean functions to interpretations by applying coordinate-wise the original function. So, if $m_1, \ldots m_k \in \{0,1\}^n$, then $\beta(m_1, \ldots, m_k)$ is defined by $(\beta(m_1[1], \ldots, m_k[1]), \ldots, \beta(m_1[n], \ldots, m_k[n]))$, where $m[i]$ is the i-th coordinate of the interpretation m.

The next definition gives a general formal definition of closure.

Definition 1. *Given a set $\mathcal{M} \subseteq 2^{\mathcal{U}}$ of interpretations and $\beta \in \mathcal{B}$, we define $Cl_\beta(\mathcal{M})$, the* closure *of $\mathcal{M}$ under β, as the smallest set of interpretations that contains $\mathcal{M}$ and that is closed under β, i.e., if $m_1, \ldots, m_k \in Cl_\beta(\mathcal{M})$, then $\beta(m_1, \ldots, m_k) \in Cl_\beta(\mathcal{M})$.*

Definition 2. *Let $\beta \in \mathcal{B}$. A set $\mathcal{L}' \subseteq \mathcal{L}$ of propositional formulas is a β-*fragment *if:*

1. *for all $\psi \in \mathcal{L}'$, $\mathrm{Mod}(\psi) = Cl_\beta(\mathrm{Mod}(\psi))$*
2. *for all $\mathcal{M} \subseteq 2^{\mathcal{U}}$ with $\mathcal{M} = Cl_\beta(\mathcal{M})$ there exists a $\psi \in \mathcal{L}'$ with $\mathrm{Mod}(\psi) = \mathcal{M}$*
3. *if $\phi, \psi \in \mathcal{L}'$ then $\phi \wedge \psi \in \mathcal{L}'$.*

We call fragments $\mathcal{L}' \subseteq \mathcal{L}$ which are β-fragments for a $\beta \in \mathcal{B}$ also characterizable fragments *(of propositional logic).*

Well-known fragments of propositional logic are $\mathcal{L}_{Horn}$ which is an $\wedge$-fragment, $\mathcal{L}_{Krom}$ which is a maj_3-fragment and $\mathcal{L}_{Affine}$ which is $\oplus_3$-fragment [16, 23]. More generally such fragments were systematically investigated in [4].

2.3 Model-based contraction

Belief contraction consists in removing a belief from an agent's belief set (theory). More formally, in model-based approaches a belief set is represented by a formula, and a contraction operator, denoted by $-$, is a function from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$ that maps two formulas ψ (the initial agent's beliefs) and μ (the belief to be removed) to a new formula $\psi - \mu$ (the contracted agent's beliefs). We recall the KM postulates for belief contraction [18].

(C1) $\psi \models \psi - \mu$
(C2) If $\psi \not\models \mu$, then $\psi - \mu \models \psi$
(C3) If $\psi - \mu \models \mu$, then $\models \mu$
(C4*) $(\psi - \mu) \wedge \mu \models \psi$
(C5) If $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$, then $\psi_1 - \mu_1 \equiv \psi_2 - \mu_2$

(C1) ensures that after contraction, no new information was added to the initial agent's beliefs. (C2) expresses that if μ is not deducible from ψ, then no change is made by the contraction of the initial agent's beliefs. (C3) guarantees that the only possibility for the contraction of ψ by μ to fail is that μ is a tautology. (C4*) says that the initial belief set ψ is deducible from the conjunction of the result of the contraction of ψ by μ and from μ. (C5) reflects the principle of independence of syntax.

More recently Caridroit, Konieczny and Marquis [3] reformulated (C4*) and proposed two new postulates (C6) and (C7):
(C4) If $\psi \models \mu$, then$(\psi - \mu) \wedge \mu \models \psi$
(C6) $\psi - (\mu_1 \wedge \mu_2) \models (\psi - \mu_1) \vee (\psi - \mu_2)$
(C7) If $\psi - (\mu_1 \wedge \mu_2) \not\models \mu_1$, then $\psi - \mu_1 \models \psi - (\mu_1 \wedge \mu_2)$

(C6) and (C7) express the minimality of change for the conjunction. (C6) says that the contraction by a conjunction always implies the disjunction of the two contractions by the conjuncts. (C7) says that if μ_1 has not been removed during the contraction by $\mu_1 \wedge \mu_2$, then the contraction by μ_1 must imply the contraction by the conjunction.

Caridroit, Konieczny and Marquis [3] proposed a representation theorem for model-based contraction operators in the same spirit as Katsuno and Mendelzon's representation theorem for revision. This theorem uses the notion of *faithful assignment* [17] which is a function that maps a formula ψ, to a pre-order $\leq_\psi$ on the interpretations as follows:

- If $m_1 \in \mathrm{Mod}(\psi)$ and $m_2 \in \mathrm{Mod}(\psi)$, then $m_1 =_\psi m_2$,
- If $m_1 \in \mathrm{Mod}(\psi)$ and $m_2 \notin \mathrm{Mod}(\psi)$, then $m_1 <_\psi m_2$,
- If $\psi_1 \equiv \psi_2$, then $\leq_{\psi_1} = \leq_{\psi_2}$.

Proposition 1. [3] *A contraction operator $-$ satisfies* (C1)-(C7) *if and only if there exists a faithful assignment that maps each formula ψ to a total preorder $\leq_\psi$ such that* $\mathrm{Mod}(\psi - \mu) = \mathrm{Mod}(\psi) \cup min(\mathrm{Mod}(\neg\mu), \leq_\psi)$.

One can define model-based contraction operators from model-based revision operators using Harper's identity [15] $\psi - \mu \equiv \psi \vee (\psi \circ \neg\mu)$. We thus define two model-based contraction operators from well-known revision operators, namely, Dalal's [7] and Satoh's [22] revision operators.

In model-based revision operators the closeness between models relies on the symmetric difference between models, that is the set of propositional variables on which they differ.

Dalal measures the minimal change by the cardinality of model change, Let ψ and μ be two propositional formulas and m and m' be two interpretations, $m \Delta m'$ denotes the symmetric difference between m and m' and $|\Delta|^{min}(\psi, \mu)$ denotes the minimum number of propositional variables on which the models of ψ and μ differ and is defined as $min\{|m \Delta m'| : m \in \mathrm{Mod}(\psi), m' \in \mathrm{Mod}(\mu)\}$. The Dalal revision operator [7], denoted by $\circ_D$, is then defined by: $\mathrm{Mod}(\psi \circ_D \mu) = \{m \in \mathrm{Mod}(\mu) : \exists m' \in \mathrm{Mod}(\psi)\ s.t.\ |m \Delta m'| = |\Delta|^{min}(\psi, \mu)\}$.

Satoh interprets the minimal change in terms of set inclusion instead of cardinality on model difference. More formally, let

$\Delta^{min}(\psi,\mu) = min_{\subseteq}\{m\Delta m' : m \in \mathrm{Mod}(\psi),\ m' \in \mathrm{Mod}(\mu)\}$. The Satoh revision operator [22], denoted by $\circ_S$, is then defined by: $\mathrm{Mod}(\psi \circ_S \mu) = \{m \in \mathrm{Mod}(\mu) : \exists m' \in \mathrm{Mod}(\psi)\ s.t.\ m\Delta m' \in \Delta^{min}(\psi,\mu)\}$.

The contraction operator obtained from Dalal's revision operator in using Harper's identity is denoted by $-_D$, and is defined by $\mathrm{Mod}(\psi -_D \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \circ_D \neg\mu)$. The contraction operator obtained from Satoh's revision operator is denoted by $-_S$, and is defined by $\mathrm{Mod}(\psi -_S \mu) = \mathrm{Mod}(\psi) \cup \mathrm{Mod}(\psi \circ_S \neg\mu)$.

Contraction operator $-_D$ satisfies (C1) − (C7) [3] while contraction operator $-_S$ satisfies (C1) − (C6), but violates (C7) [19].

3 REFINEMENT OF BELIEF CHANGE OPERATORS

The problem of standard belief change operators when applied in a fragment of propositional logic is illustrated in the following example, in the case of contraction.

Example 1. *Let ψ and μ be two Horn formulas such that* $\mathrm{Mod}(\psi) = \{\emptyset, \{a\}, \{b\}\}$ *and* $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$ *(such formulas exist since these sets of models are closed under $\wedge$). Note that* $\mathrm{Mod}(\neg\mu) = \{\{a,b\},\{a,c\},\{b,c\},\{a,b,c\}\}$. *The result of the contraction of ψ by μ using Satoh's or Dalal's operator can be easily read in the following table, in which the distance between each model of ψ and each model of $\neg\mu$ is indicated.*

$Mod(\psi)$	$Mod(\neg\mu)$			
	{a,b}	{a,c}	{b,c}	{a,b,c}
{a}	***1*** {***b***}	***1*** {***c***}	3 {a,b,c}	2 {b,c}
{b}	***1*** {***a***}	3 {a,b,c}	***1*** {***c***}	2 {a,c}
$\emptyset$	2 {a,b}	2 {a,c}	2 {b,c}	3 {a,b,c}

Therefore, $\mathrm{Mod}(\psi -_S \mu) = \mathrm{Mod}(\psi -_D \mu) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$. *This set is not $\wedge$-closed ($\{c\}$ is missing), therefore there is no formula in $\mathcal{L}_{Horn}$ that has this set of models.*

In this example, in order to adapt $-_S$ (or likewise $-_D$) so that the outcome of the contraction is in $\mathcal{L}_{Horn}$ we have several options: one is to build the closure of the set of models, in our case we have to add $\{c\}$; or to remove either $\{a,c\}$ or $\{b,c\}$ or both.

The considerations of the above example, originally studied in the context of revision in [6], can be generalized to the following problem statement: given a belief change operator $\triangle : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ and a fragment $\mathcal{L}'$ of propositional logic, how can $\triangle$ be adapted (or *refined*) to a new operator $\blacktriangle$ such that for all $\psi,\mu \in \mathcal{L}'$, also $\psi\blacktriangle\mu \in \mathcal{L}'$?

As proposed in [6] few natural desiderata for such refined operators can be stated.

Definition 3. *Let $\mathcal{L}'$ be a fragment of propositional logic and $\triangle$: $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$ a belief change operator. We call an operator $\blacktriangle$: $\mathcal{L}' \times \mathcal{L}' \to \mathcal{L}'$ a $\triangle$-refinement for $\mathcal{L}'$ if it satisfies the following properties, for each $\psi, \psi', \mu, \mu' \in \mathcal{L}'$:*

(i) Consistency: *$\psi\blacktriangle\mu$ is satisfiable if and only if $\psi \triangle \mu$ is satisfiable.*
(ii) Equivalence: *If $\psi \triangle \mu \equiv \psi' \triangle \mu'$, then $\psi\blacktriangle\mu \equiv \psi'\blacktriangle\mu'$.*
(iii) Containment: *$T_{\mathcal{L}'}(\psi \triangle \mu) \subseteq T_{\mathcal{L}'}(\psi\blacktriangle\mu)$.*
(iv) Invariance: *If $\psi \triangle \mu \in \mathcal{L}'$, then $T_{\mathcal{L}'}(\psi\blacktriangle\mu) = T_{\mathcal{L}'}(\psi \triangle \mu)$.*

In [6] the authors defined such refined operators in the context of revision through the notion of β-mappings as defined below. This can be generalized to any belief change operator operating from $\mathcal{L} \times \mathcal{L}$ to $\mathcal{L}$.

Definition 4. *Given $\beta \in \mathcal{B}$, we define a β-*mapping*, f_β, as an application from sets of models into sets of models, $f_\beta : 2^{2^{\mathcal{U}}} \longrightarrow 2^{2^{\mathcal{U}}}$, such that for every $\mathcal{M} \subseteq 2^{\mathcal{U}}$:*

1. *$Cl_\beta(f_\beta(\mathcal{M})) = f_\beta(\mathcal{M})$, i.e., $f_\beta(\mathcal{M})$ is closed under β.*
2. *$f_\beta(\mathcal{M}) \subseteq Cl_\beta(\mathcal{M})$.*
3. *If $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}) = \mathcal{M}$.*
4. *If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}) \neq \emptyset$.*

Starting from well-known belief change operators we can define new belief change operators adapted to any fragment of propositional logic $\mathcal{L}'$ in using β-mappings.

Definition 5. *Let $\triangle$: $\mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ be a belief change operator and $\mathcal{L}' \subseteq \mathcal{L}$ a β-fragment of classical logic with $\beta \in \mathcal{B}$. Given a β-mapping f_β, we denote with $\triangle^{f_\beta}$: $\mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ the operator for $\mathcal{L}'$ defined as $\mathrm{Mod}(\psi \triangle^{f_\beta} \mu) := f_\beta(\mathrm{Mod}(\psi \triangle \mu))$. The class $[\triangle, \mathcal{L}']$ contains all operators $\triangle^{f_\beta}$ where f_β is a β-mapping.*

Interestingly and as in [6], this class actually captures all refinements we had in mind.

Proposition 2. *Let $\triangle$: $\mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ be a belief change operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of classical logic. Then, $[\triangle, \mathcal{L}']$ is the set of all $\triangle$-refinements for $\mathcal{L}'$.*

Proof. A similar result was obtained in [6] for basic (revision) operators, i.e., operators satisfying $\top \triangle \mu \equiv \mu$. This assumption was only used to prove that any $\triangle$-refinement can be defined through a β-mapping. We give here an alternative proof that does not rely on this assumption.

Let $\blacktriangle$ be a $\triangle$-refinement for $\mathcal{L}'$. We show that $\blacktriangle \in [\triangle, \mathcal{L}']$. Let f be defined as follows for any set $\mathcal{M}$ of interpretations: $f(\emptyset) = \emptyset$ and for $\mathcal{M} \neq \emptyset$, if there exists a pair $(\psi_{\mathcal{M}}, \mu_{\mathcal{M}})$ of formulas from $\mathcal{L}'$ such that $\mathrm{Mod}(\psi_{\mathcal{M}} \triangle \mu_{\mathcal{M}}) = \mathcal{M}$, then we define $f(\mathcal{M}) = \mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}})$, otherwise $f(\mathcal{M}) = Cl_\beta(\mathcal{M})$. Thus the refined operator $\blacktriangle$ behaves like the operator $\triangle^f$.

We show that such a mapping f is a β-mapping. Note that since $\blacktriangle$ is a β-refinement, it satisfies the property of equivalence, thus the actual choice of the pair $(\psi_{\mathcal{M}}, \mu_{\mathcal{M}})$ is not relevant, i.e., given $\mathcal{M}$, and pairs $(\psi_{\mathcal{M}}, \mu_{\mathcal{M}})$, $(\psi'_{\mathcal{M}}, \mu'_{\mathcal{M}})$ such that $\mathrm{Mod}(\psi_{\mathcal{M}} \triangle \mu_{\mathcal{M}}) = \mathrm{Mod}(\psi'_{\mathcal{M}} \triangle \mu'_{\mathcal{M}}) = \mathcal{M}$, we have that $\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}}$ is equivalent to $\psi'_{\mathcal{M}}\blacktriangle\mu'_{\mathcal{M}}$. Thus f is well-defined.

We continue to show that the four properties in Definition 4 hold for f. Property 1 is ensured since for every $\mathcal{M}$, $f(\mathcal{M})$ is closed under β. Indeed, either $f(\mathcal{M}) = \mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}})$ and since $\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}} \in \mathcal{L}'$ its set of models is closed under β, or $f(\mathcal{M}) = Cl_\beta(\mathcal{M})$. Let us show Property 2 , i.e., $f(\mathcal{M}) \subseteq Cl_\beta(\mathcal{M})$ for any set of interpretations $\mathcal{M}$. It is obvious when $\mathcal{M} = \emptyset$ (then $f(\mathcal{M}) = \emptyset$), as well as when $f(\mathcal{M}) = Cl_\beta(\mathcal{M})$. Otherwise $f(\mathcal{M}) = \mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}})$ and since $\blacktriangle$ satisfies containment $\mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}}) \subseteq Cl_\beta(\mathrm{Mod}(\psi_{\mathcal{M}} \triangle \mu_{\mathcal{M}})$. Therefore in any case we have $f(\mathcal{M}) \subseteq Cl_\beta(\mathcal{M})$. For showing Property 3 let us consider $\mathcal{M} \neq \emptyset$ such that $\mathcal{M} = Cl_\beta(\mathcal{M})$. If $f(\mathcal{M}) = Cl_\beta(\mathcal{M})$, then $f(\mathcal{M}) = \mathcal{M}$. Otherwise, $f(\mathcal{M}) = \mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}})$ where $\psi_{\mathcal{M}}, \mu_{\mathcal{M}} \in \mathcal{L}'$ such that $\mathrm{Mod}(\psi_{\mathcal{M}} \triangle \mu_{\mathcal{M}}) = \mathcal{M}$. Since $\blacktriangle$ satisfies invariance $\mathrm{Mod}(\psi_{\mathcal{M}}\blacktriangle\mu_{\mathcal{M}}) = \mathcal{M}$. Thus, in any case, $f(\mathcal{M}) = \mathcal{M}$. Property 4 is ensured by consistency of $\blacktriangle$. □

Hence, β-mappings will allow us to define a variety of refined operators. We will consider two β-mappings in particular, namely the closure Cl_β defined above and Min_β defined below.

Definition 6. *Let $\beta \in \mathcal{B}$ and suppose that $\leq$ is a fixed linear order on the set $2^{\mathcal{U}}$ of interpretations. We define the function Min_β as $\mathrm{Min}_\beta(\mathcal{M}) = \mathcal{M}$ if $Cl_\beta(\mathcal{M}) = \mathcal{M}$, and $\mathrm{Min}_\beta(\mathcal{M}) = Min_{\leq}(\mathcal{M})$ otherwise.*

For $\mathcal{L}'$ a β-fragment and $\triangle$ an operator, the corresponding operators $\triangle^{Cl_\beta}$ and $\triangle^{\mathrm{Min}_\beta}$ are thus respectively given as $\mathrm{Mod}(\psi \triangle^{Cl_\beta} \mu) = Cl_\beta(\mathrm{Mod}(\psi \triangle \mu))$ and $\mathrm{Mod}(\psi \triangle^{\mathrm{Min}_\beta} \mu) = \mathrm{Min}_\beta(\mathrm{Mod}(\psi \triangle \mu))$.

Example 2. *Recall Example 1 where we had $\psi, \mu \in \mathcal{L}_{Horn}$ with $\mathrm{Mod}(\psi - \mu) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}\}$ ($- \in \{-_D, -_S\}$). Our refined operator $-^{Cl_\wedge}$ provides*

$$\mathrm{Mod}(\psi -^{Cl_\wedge} \mu) = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{b,c\}, \{c\}\}.$$

Assume that we have the following order, $\leq$ on the set of interpretations $\emptyset < \{a\} < \{b\} < \{c\} < \{a,b\} < \{a,c\} < \{b,c\} < \{a,b,c\}$. Then the refined operator $-^{\mathrm{Min}_\wedge}$ provides

$$\mathrm{Mod}(\psi -^{\mathrm{Min}_\wedge} \mu) = \{\emptyset\}.$$

A natural objective is now to study how refined belief change operators behave with respect to satisfaction of postulates that characterize rational operators. This has been already done for revision operators [6], as well as for update operators [5]. We aim at doing it for contraction.

Remind that Harper's identity allows one to define model-based contraction operators from model-based revision operators, $\psi - \mu \equiv \psi \vee (\psi \circ \neg\mu)$. Nevertheless this identity does not allow one to obtain a contraction operator that is adapted to a fragment from a revision operator that is so. Indeed, this identity makes first the revision operator act on the negation of a formula, and second consider the disjunction of two formulas. However characterizable fragments are neither closed under negation nor under disjunction (i.e., given two formulas μ_1 and μ_2 in a β-fragment $\mathcal{L}'$, neither $\neg\mu_1$, nor $\mu_1 \vee \mu_2$ is necessarily equivalent to a formula in $\mathcal{L}'$).

So, in order to obtain contraction operators that are adapted to fragments it makes sense to study refinements of usual contraction operators. This is what we do in the next section.

4 REFINEMENT OF CONTRACTION OPERATORS

The characterization of refined operators gives a way to define concrete refined operators for which we can study the satisfaction of rationality postulates. The property of containment for a refinement (property *(iii)* in Definition 3) guarantees that the refined operator approximates the original operator, in the sense that the refinement preserves the logical consequences of the original operator within the considered fragment. In the context of revision this property ensures in particular that if μ is a logical consequence of the revision $\psi \circ \mu$, then μ is also a logical consequence of the refined revision $\psi \bullet \mu$. Hence, this property contributes to the preservation of basic postulates when refining revision operators. In contrast, it turns out to be insufficient in the case of contraction.

We say that a contraction operator satisfies a KM postulate $(\mathrm{C_i})$ $(i = 1, \ldots, 7)$ in $\mathcal{L}'$ if the respective postulate holds when restricted to formulas in $\mathcal{L}'$.

4.1 Reasonable refinements

In this section we first show a positive result concerning the preservation of two basic KM postulates by refinement of contraction operators.

Proposition 3. *Let $-$ be a contraction operator satisfying KM postulate (C2) (resp., (C5)) and $\mathcal{L}' \subseteq \mathcal{L}$ be a characterizable fragment. Then each refinement of this operator $\circledast \in [-, \mathcal{L}']$ satisfies (C2) (resp., (C5)) in $\mathcal{L}'$ as well.*

Proof. Since $\mathcal{L}'$ a characterizable fragment, $\mathcal{L}'$ is a β-fragment for some $\beta \in \mathcal{B}$. According to Proposition 2 we can assume that $\circledast \in [-, \mathcal{L}']$ is an operator of the form $-^{f_\beta}$, where f_β is a suitable β-mapping. We show that $-^{f_\beta}$ satisfies (C2) and (C5) for all ψ and $\mu \in \mathcal{L}'$.
(C2) states that if $\psi \not\models \mu$, then $\mathrm{Mod}(\psi - \mu) \subseteq \mathrm{Mod}(\psi)$. Assume that $\psi \not\models \mu$. Since $-$ satisfies (C2), then $\mathrm{Mod}(\psi - \mu) \subseteq \mathrm{Mod}(\psi)$. Thus $Cl_\beta(\mathrm{Mod}(\psi - \mu)) \subseteq Cl_\beta(\mathrm{Mod}(\psi))$ by monotonicity of the closure. Hence, $Cl_\beta(\mathrm{Mod}(\psi - \mu)) \subseteq \mathrm{Mod}(\psi)$ since $\psi \in \mathcal{L}'$ and $\mathcal{L}'$ is a β-fragment. According to property 2 in Definition 4 $f_\beta(\mathrm{Mod}(\psi - \mu)) \subseteq Cl_\beta(\mathrm{Mod}(\psi - \mu))$, hence $f_\beta(\mathrm{Mod}(\psi - \mu)) \subseteq \mathrm{Mod}(\psi)$. By definition of $\circledast$, this means that $\psi \circledast \mu \models \psi$.
(C5) : Let ψ_1, ψ_2, μ_1 and μ_2 in $\mathcal{L}'$ such that $\psi_1 \equiv \psi_2$ and $\mu_1 \equiv \mu_2$. Since $-$ satisfies (C5), $\psi_1 - \mu_1 \equiv \psi_2 - \mu_2$. Since $\circledast$ is a $-$-refinement, $\psi_1 \circledast \mu_1 \equiv \psi_2 \circledast \mu_2$ by the property of equivalence (Definition 3). □

In contrast postulates (C1) and (C3) are not preserved by all refinements as illustrated by the following proposition.

Proposition 4. *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then the refined operator $-^{\mathrm{Min}_\beta}$ violates postulates (C1) and (C3) in $\mathcal{L}'$.*

Proof. Example 2 gives two formulas ψ and μ in $\mathcal{L}_{Horn}$ such that on the one hand $\mathrm{Mod}(\psi) \not\subseteq \mathrm{Mod}(\psi -^{Min_\wedge} \mu)$, and on the other hand $\mathrm{Mod}(\psi -^{Min_\wedge} \mu) \subseteq \mathrm{Mod}(\mu)$ but μ is not a tautology. Therefore it proves the proposition in $\mathcal{L}_{Horn}$. Actually, it also proves it in the case of $\mathcal{L}_{Krom}$ since it is easily seen that the given sets of models are also closed under maj_3, and therefore there exist formulas in $\mathcal{L}_{Krom}$ having these sets of models as well. □

In conclusion, in the context of contraction, while the notion of refinement continues to express a kind of approximation of the original operator, it fails at preserving all basic postulates, in particular (C1) and (C3). Thus, refined contraction operators will not necessarily behave rationally. To overcome this difficulty we have to restrict refinements to *reasonable* ones, which are refinements having two additional properties.

Definition 7. *Let $\mathcal{L}'$ be a fragment of propositional logic and $- : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ a contraction operator. We call an operator $\circledast : \mathcal{L}' \times \mathcal{L}' \to \mathcal{L}'$ a* $-$-reasonable refinement for *$\mathcal{L}'$ if it is a $-$-refinement that satisfies in addition the two following properties. For all ψ, ψ', μ and $\mu' \in \mathcal{L}'$,*

- *(v) : If $T_{\mathcal{L}}(\psi - \mu) \subseteq T_{\mathcal{L}}(\psi)$, then $T_{\mathcal{L}'}(\psi \circledast \mu) \subseteq T_{\mathcal{L}'}(\psi)$.*
- *(vi) : If $T_{\mathcal{L}}(\mu) \not\subseteq T_{\mathcal{L}}(\psi - \mu)$, then $T_{\mathcal{L}'}(\mu) \not\subseteq T_{\mathcal{L}'}(\psi \circledast \mu)$.*

Property *(v)* states that if no new information is added to the initial agent's beliefs by the original operator, then none is either by the refined operator. Property *(vi)* means that if μ is not deducible from the result of the contraction $\psi - \mu$ by the original operator, then it is not either from the result of the contraction $\psi \circledast \mu$ by the refined operator.

The refinement by the closure is such a reasonable refinement.

Proposition 5. *For any contraction operator $- : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and any β-fragment $\mathcal{L}' \subseteq \mathcal{L}$ of classical logic, $-^{Cl_\beta}$ is a reasonable $-$-refinement for $\mathcal{L}'$.*

Proof. The operator $-^{Cl_\beta}$ is a $-$-refinement for $\mathcal{L}'$, it remains to show that it is a reasonable one, i.e., that it verifies properties *(v)* and *(vi)* in Definition 7.

(v) : Suppose that $T_\mathcal{L}(\psi - \mu) \subseteq T_\mathcal{L}(\psi)$, that is $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\psi - \mu)$. By monotonicity, $Cl_\beta(\mathrm{Mod}(\psi)) \subseteq Cl_\beta(\mathrm{Mod}(\psi - \mu))$. Since $\psi \in \mathcal{L}'$, we thus get $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\psi -^{Cl_\beta} \mu)$, hence $T_{\mathcal{L}'}(\psi -^{Cl_\beta} \mu) \subseteq T_{\mathcal{L}'}(\psi)$.

(vi) : Suppose that $T_\mathcal{L}(\mu) \nsubseteq T_\mathcal{L}(\psi - \mu)$. Then, $\mathrm{Mod}(\psi - \mu) \nsubseteq \mathrm{Mod}(\mu)$, and *a fortiori* $Cl_\beta(\mathrm{Mod}(\psi - \mu)) \nsubseteq \mathrm{Mod}(\mu)$, i.e. $\mathrm{Mod}(\psi -^{Cl_\beta} \mu) \nsubseteq \mathrm{Mod}(\mu)$. Since μ is in $\mathcal{L}'$, it follows that $T_{\mathcal{L}'}(\mu) \nsubseteq T_{\mathcal{L}'}(\psi -^{Cl_\beta} \mu)$. □

We now show how to characterize all reasonable refinements.

4.2 Characterization of reasonable refinements

The characterization of all reasonable refinements of a contraction operator within a fragment uses the notion of β-contract-mapping defined as follows.

Definition 8. *Given $\beta \in \mathcal{B}$, we define a β-*contract-mapping *, f_β, as an application f_β: $2^{2^\mathcal{U}} \times 2^{2^\mathcal{U}} \times 2^{2^\mathcal{U}} \longrightarrow 2^{2^\mathcal{U}}$, such that for all sets of models $\mathcal{M}$, $\mathcal{M}_1$, $\mathcal{M}_2$ in $2^{2^\mathcal{U}}$:*

1. *$Cl_\beta(f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)) = f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$,*
2. *$f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \subseteq Cl_\beta(\mathcal{M})$,*
3. *If $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \mathcal{M}$,*
4. *If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \neq \emptyset$,*
5. *If $\mathcal{M}_1 \subseteq \mathcal{M}$, then $\mathcal{M}_1 \subseteq f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$,*
6. *If $\mathcal{M} \nsubseteq \mathcal{M}_2$, then $f_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \nsubseteq \mathcal{M}_2$.*

Observe that by abuse of notation the application Cl_β can be defined by $Cl_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$. It is then easy to verify that this application satisfies all properties of Definition 8 and thus is a β-contract-mapping. The concept of contract-mapping allows us to define a family of reasonable refined operators for fragments of propositional logic as follows.

Definition 9. *Let $- : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ be a contraction operator and $\mathcal{L}' \subseteq \mathcal{L}$ a β-fragment of classical logic with $\beta \in \mathcal{B}$. For a β-contract-mapping, f_β, we denote with $-^{f_\beta} : \mathcal{L}' \times \mathcal{L}' \longrightarrow \mathcal{L}'$ the operator for $\mathcal{L}'$ defined as*

$$\mathrm{Mod}(\psi -^{f_\beta} \mu) := f_\beta(\mathrm{Mod}(\psi - \mu), \mathrm{Mod}(\psi), \mathrm{Mod}(\mu)).$$

The class $\langle -, \mathcal{L}' \rangle$ contains all operators $-^{f_\beta}$ where f_β is a β-contract-mapping.

The next proposition reflects that the above class captures all reasonable refined contraction operators we had in mind.

Proposition 6. *Let $- : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ be a contraction operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of propositional logic. Then, $\langle -, \mathcal{L}' \rangle$ is the set of all reasonable $-$-refinements for $\mathcal{L}'$.*

Proof. Let us first show that any operator $-^{f_\beta} \in \langle -, \mathcal{L}' \rangle$ is a reasonable $-$-refinement for $\mathcal{L}'$. Observe that while a β–contract-mapping is a ternary application, the first four properties defining it depend only on the first variable and coincide with the properties of a β-mapping. Therefore, according to Proposition 2 the operator $-^{f_\beta}$ is $-$-refinement for $\mathcal{L}'$. We only have to prove that it satisfies the two additional properties in Definition 7.

(v) Suppose that $T_\mathcal{L}(\psi - \mu) \subseteq T_\mathcal{L}(\psi)$. Then, $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\psi - \mu)$ and according to property 5 in Definition 8, $\mathrm{Mod}(\psi) \subseteq f_\beta(\mathrm{Mod}(\psi - \mu), \mathrm{Mod}(\psi), \mathrm{Mod}(\mu))$, i.e., $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\psi -^{f_\beta} \mu)$. Hence, $T_\mathcal{L}(\mu) \subseteq T_\mathcal{L}(\psi -^{f_\beta} \mu)$, and *a fortiori* $T_{\mathcal{L}'}(\mu) \subseteq T_{\mathcal{L}'}(\psi -^{f_\beta} \mu)$.

(vi) Suppose that $T_\mathcal{L}(\mu) \nsubseteq T_\mathcal{L}(\psi - \mu)$. Then, $\mathrm{Mod}(\psi - \mu) \nsubseteq \mathrm{Mod}(\mu)$ and according to property 6 in Definition 8, $f_\beta(\mathrm{Mod}(\psi - \mu), \mathrm{Mod}(\psi), \mathrm{Mod}(\mu)) \nsubseteq \mathrm{Mod}(\mu)$, i.e., $\mathrm{Mod}(\psi -^{f_\beta} \mu) \nsubseteq \mathrm{Mod}(\mu)$. Hence, $\mathrm{Mod}(\psi -^{f_\beta} \mu) \nsubseteq Cl_\beta(\mathrm{Mod}(\mu))$ and since $\mu \in \mathcal{L}'$ $T_{\mathcal{L}'}(\mu) \nsubseteq T_{\mathcal{L}'}(\psi -^{f_\beta} \mu)$.

Conversely, given $\circledast$ a reasonable $-$-refinement for $\mathcal{L}'$. Let us prove that $\circledast \in \langle -, \mathcal{L}' \rangle$. Consider the application f defined for all triple of sets of interpretations $(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$ as follows. If $\mathcal{M} = \emptyset$, then $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \emptyset$. If $\mathcal{M} \neq \emptyset$ and if there exists a pair of formulas $(\psi_\mathcal{M}, \mu_\mathcal{M})$ in $\mathcal{L}'$, such that $\mathrm{Mod}(\psi_\mathcal{M} - \mu_\mathcal{M}) = \mathcal{M}$ and $\mathrm{Mod}(\psi_\mathcal{M}) = \mathcal{M}_1$ and $\mathrm{Mod}(\mu_\mathcal{M}) = \mathcal{M}_2$, then we defined $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \mathrm{Mod}(\psi_\mathcal{M} \circledast \mu_\mathcal{M})$. Otherwise $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$.

First observe that this application is well defined. Indeed, since the operator $\circledast$ is a reasonable $-$-refinement for $\mathcal{L}'$, it does not depend on the choice of the pair $(\psi_\mathcal{M}, \mu_\mathcal{M})$. Moreover, this application satisfies the first four properties in Definition 8. We have to verify the last two ones.

(5) Suppose that $\mathcal{M}_1 \subseteq \mathcal{M}$ (the case where $\mathcal{M} = \emptyset$ is trivial). If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$, then $\mathcal{M}_1 \subseteq f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$. Now, let us turn to the case where $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \mathrm{Mod}(\psi_\mathcal{M} \circledast \mu_\mathcal{M})$, with $\mathcal{M} = \mathrm{Mod}(\psi - \mu)$, $\mathcal{M}_1 = \mathrm{Mod}(\psi)$ and $\mathcal{M}_2 = \mathrm{Mod}(\mu)$. Since $\mathcal{M}_1 \subseteq \mathcal{M}$, $T_\mathcal{L}(\mathcal{M}) \subseteq T_\mathcal{L}(\mathcal{M}_1)$, that is $T_\mathcal{L}(\psi_\mathcal{M} - \mu_\mathcal{M}) \subseteq T_\mathcal{L}(\psi_\mathcal{M})$. Since $\circledast$ satisfies property (v) in Definition 7, we get $T_{\mathcal{L}'}(\psi_\mathcal{M} \circledast \mu_M) \subseteq T_{\mathcal{L}'}(\psi_\mathcal{M})$. Therefore $\mathrm{Mod}(\psi_\mathcal{M}) \subseteq \mathrm{Mod}(\psi_\mathcal{M} \circledast \mu_\mathcal{M})$ since $\psi_\mathcal{M} \circledast \mu_\mathcal{M} \in \mathcal{L}'$. This proves that $\mathcal{M}_1 \subseteq f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2)$.

(6) Suppose that $\mathcal{M} \nsubseteq \mathcal{M}_2$. If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = Cl_\beta(\mathcal{M})$, then $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \nsubseteq \mathcal{M}_2$. If $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) = \mathrm{Mod}(\psi_\mathcal{M} \circledast \mu_\mathcal{M})$, with $\mathcal{M} = \mathrm{Mod}(\psi_\mathcal{M} - \mu_\mathcal{M})$, $\mathcal{M}_1 = \mathrm{Mod}(\psi_\mathcal{M})$ and $\mathcal{M}_2 = \mathrm{Mod}(\mu_\mathcal{M})$, then $\mathrm{Mod}(\psi_\mathcal{M} - \mu_\mathcal{M}) \nsubseteq \mathrm{Mod}(\mu_\mathcal{M})$, i.e., $T_\mathcal{L}(\mu_\mathcal{M}) \nsubseteq T_\mathcal{L}(\psi_\mathcal{M} - \mu_\mathcal{M})$. Therefore, according to property (vi) in Definition 7, we get $T_{\mathcal{L}'}(\mu_\mathcal{M}) \nsubseteq T_{\mathcal{L}'}(\psi_\mathcal{M} \circledast \mu_\mathcal{M})$. Therefore, $\mathrm{Mod}(\psi_\mathcal{M} \circledast \mu_\mathcal{M}) \nsubseteq \mathrm{Mod}(\mu_\mathcal{M})$ since $\mu_\mathcal{M}$ is in $\mathcal{L}'$. This proves that $f(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) \nsubseteq \mathcal{M}_2$.

□

So far we have considered $-^{Cl_\beta}$ as one instantiation of a reasonable $-$-refinement. In order to get further concrete reasonable refinements we need to define further β-contract-mappings. An additional example is as follows.

Definition 10. *Let $\beta \in \mathcal{B}$ and suppose that $\leq$ is a total order on the set $2^\mathcal{U}$ of interpretations. We define the function p_β as*

$p_\beta(\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2) =$

$$\begin{cases} \mathcal{M} & \text{if } \mathcal{M} = Cl_\beta(\mathcal{M}) \\ Cl_\beta(\mathcal{M}_1 \cup Min_\leq(\mathcal{M} \cap \overline{\mathcal{M}_2})) & \text{else and if } \mathcal{M}_1 \subseteq \mathcal{M} \\ & \text{and } \mathcal{M} \cap \overline{\mathcal{M}_2} \neq \emptyset \\ Cl_\beta(\mathcal{M}) & \text{otherwise} \end{cases}$$

It is easy to verify that the function p_β satisfies all six properties in Definition 8. As such p_β is a β-contract-mapping. Therefore, according to Proposition 6, for $\mathcal{L}'$ a β-fragment and $-$ a contraction operator, it holds that the operator $-^{p_\beta}$ defined as

$$\text{Mod}(\psi -^{p_\beta} \mu) = p_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu))$$

is a reasonable $-$-refinement for $\mathcal{L}'$.

Example 3. *Recall Example 1 where we had $\psi, \mu \in \mathcal{L}_{Horn}$ with $\text{Mod}(\psi) = \{\emptyset, \{a\}, \{b\}\}$, $\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, and $\text{Mod}(\psi - \mu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Suppose that we have the following order, $\leq$, on the set of interpretations $\emptyset < \{a\} < \{b\} < \{c\} < \{a, b\} < \{a, c\} < \{b, c\} < \{a, b, c\}$.*

Our refined operator $-^{p_\wedge}$ provides $\text{Mod}(\psi -^{p_\wedge} \mu) = Cl_\wedge(\{\emptyset, \{a\}, \{b\}\} \cup Min_\leq\{\{a, b\}, \{a, c\}, \{b, c\}\}$, that is $\text{Mod}(\psi -^{p_\wedge} \mu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

4.3 Satisfaction of postulates

In this section we study the properties of our refined contraction operators in terms of satisfaction of KM postulates.

We first show a positive result concerning four basic postulates. We prove that (C1),(C2), (C3) and (C5) are preserved by any reasonable refinement. For the other postulates we obtain more negative results. As a negative result we know that for the Horn fragment, there is no reasonable refinement of any contraction operator that satisfies (C4). We prove that the refinements of Satoh's and Dalal's contraction operators by the two mappings we consider here, Cl_β and p_β, violate (C4) in the Krom fragment as well. We get a similar negative result for the postulate (C6) in both Horn and Krom fragments. For the postulate (C7) the results are more contrasted, the refinement by closure preserves this postulate, while the p_β-refinement does not.

Proposition 7. *Let $-$ be a contraction operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. If $-$ satisfies postulate (C1),(resp. (C2), (C3) and (C5)), then so does any reasonable refinement of this operator $\circledast \in \langle -\mathcal{L}' \rangle$ in $\mathcal{L}'$.*

Proof. Since a reasonable refinement is a refinement, according to Proposition 3 we only have to prove that (C1) and (C3) are preserved. We can assume that $\circledast = -^{f_\beta}$ for some suitable β-contract-mapping f_β. Let ψ and μ two formulas in $\mathcal{L}'$.
(C1): Since $-$ satisfies (C1), $\text{Mod}(\psi) \subseteq \text{Mod}(\psi - \mu)$. According to property 5 in Definition 8, we have $\text{Mod}(\psi) \subseteq f_\beta(\text{Mod}(\psi - \mu), \text{Mod}(\psi), \text{Mod}(\mu))$, i.e., $\psi \models \psi -^{f_\beta} \mu$. So, $\psi \models \psi \circledast \mu$.
(C3): Suppose that $\psi \circledast \mu \models \mu$, i.e., $\text{Mod}(\psi -^{f_\beta} \mu) \subseteq \text{Mod}(\mu)$. According to property 6 in Definition 8, we get $\text{Mod}(\psi - \mu) \subseteq \text{Mod}(\mu)$. Since $-$ satisfies (C3), $\models \mu$ holds. □

A natural question is whether one can find reasonable refined operators for characterizable fragments that satisfy all postulates. Actually, this question has already been answered in the Horn fragment.

Indeed, starting from another perspective Flouris et al. studied belief change in a more general setting than classical logic [10]. They gave a necessary and sufficient condition for the existence of a contraction operator satisfying the basic AGM postulates in terms of decomposability. But it was shown in [20] that the Horn fragment is not decomposable. Hence it is not possible to define a Horn contraction that satisfies postulate (C4). In particular the following holds.

Proposition 8. *Let $-$ be a contraction operator. Then any reasonably refined refined operator $\circledast \in \langle -, \mathcal{L}_{Horn} \rangle$ violates postulate (C4) in $\mathcal{L}_{Horn}$.*

As far as we know, there is no such a general result for the Krom fragment. We get nevertheless a negative result for the refinement of Satoh's and Dalal's contraction operators by the two mappings we consider here, in the Krom fragment.

Proposition 9. *Let $- \in \{-_D, -_S\}$. Then $-^{Cl_{\text{maj}_3}}$ and $-^{p_{\text{maj}_3}}$ violate postulate (C4) in $\mathcal{L}_{Krom}$.*

Proof. (C4) states that if $\psi \models \mu$, then $(\psi - \mu) \wedge \mu \models \psi$. Let $- \in \{-_D, -_S\}$. By definition there is a maj_3-contract-mapping f_{maj_3} such that $\circledast = -^{f_{\text{maj}_3}}$. Consider ψ and μ in $\mathcal{L}_{Krom}$ such that $\text{Mod}(\psi) = \{\emptyset, \{a, b\}, \{c, d\}\}$ and $\text{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, c, d\}\}$. Such formulas exist since the corresponding sets of models are maj_3-closed. Observe in addition that $\psi \models \mu$. We have $\text{Mod}(\neg\mu) = \{\{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. One can easily check that $\text{Mod}(\psi - \mu) = \{\emptyset, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$. Observe that this set is not closed under maj_3. In particular $\{c\} \in Cl_{maj_3}(\text{Mod}(\psi - \mu))$. Therefore, $\{c\}$, which is a model of μ but not a model of ψ, belongs to $\text{Mod}(\psi -^{Cl_{\text{maj}_3}} \mu)$, thus proving that $(\psi -^{Cl_{\text{maj}_3}} \mu) \wedge \mu \not\models \psi$.

Assume now that we have the following order on interpretations: $\{a, b, c\} < \{a, b, d\} < \{b, c, d\}$. Then $\text{Mod}(\psi -^{p_{\text{maj}_3}} \mu) = Cl_{\text{maj}_3}(\{\emptyset, \{a, b\}, \{c, d\}\} \cup \{\{a, b, c\}\})$. Therefore, $\{c\} \in \text{Mod}(\psi -^{p_{\text{maj}_3}} \mu)$ and we conclude as above. □

We get also a negative result for postulate (C6).

Proposition 10. *Let $- \in \{-_D, -_S\}$. Then $-^{Cl_\wedge}$ and $-^{p_\wedge}$ violate postulate (C6) in $\mathcal{L}_{Horn}$, and $-^{Cl_{\text{maj}_3}}$ violates postulate (C6) in $\mathcal{L}_{Krom}$.*

Proof. Let $- \in \{-_D, -_S\}$. We first show that $-^{Cl_\beta}$ violates (C6) in $\mathcal{L}_{Horn}$. Let ψ, μ_1 and μ_2 be Horm formulas such that $\text{Mod}(\psi) = \{\{a, b, c, d\}\}$, $\text{Mod}(\mu_1) = \{\emptyset, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}\}$ and $\text{Mod}(\mu_2) = \{\emptyset, \{b\}, \{d\}, \{a, c\}, \{b, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, c, d\}\}$. We have then $\text{Mod}(\neg(\mu_1 \wedge \mu_2)) = \{\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{b, c, d\}, \{a, c, d\}\}$. On the one hand, $\text{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. This set is not closed under $\wedge$ ($\{c, d\}$ is missing). Therefore, $\text{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)) = \{\{a, b, c, d\}, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. On the other hand $\text{Mod}(\psi - \mu_1) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}\}$. This set is not closed under $\wedge$. Therefore $\text{Mod}(\psi -^{Cl_\beta} \mu_1) = \{\{a, b, c, d\}, \{b, c, d\}, \{a, c, d\}, \{c, d\}\}$. Moreover $\text{Mod}(\psi - \mu_2) = \{\{a, b, c, d\}, \{a, b, d\}, \{b, c, d\}\}$, which is not closed under $\wedge$ either ($\{b, d\}$ is missing). Therefore, $\text{Mod}(\psi -^{Cl_\beta} \mu_2) = \{\{a, b, c, d\}, \{a, b, d\}, \{b, c, d\}, \{b, d\}\}$.

Observe that $\mathrm{Mod}(\psi -^{Cl_\beta} \mu_1) \cup \mathrm{Mod}(\psi -^{Cl_\beta} \mu_2) = \{\{a,b,c,d\},\{b,c,d\},\{a,c,d\},\{a,b,d\},\{c,d\},\{b,d\}\}$. We conclude that $\mathrm{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)) \nsubseteq \mathrm{Mod}(\psi -^{Cl_\beta} \mu_1) \cup \mathrm{Mod}(\psi -^{Cl_\beta} \mu_2)$, which proves that $-^{Cl_\wedge}$ violates (C6) in $\mathcal{L}_{Horn}$.

Let us now prove that $-^{p_\beta}$ violates (C6) in $\mathcal{L}_{Horn}$. Consider ψ, μ_1 and μ_2 Horn formulas such that

$$\mathrm{Mod}(\psi) = \{\emptyset, \{a,b,c\}\},$$

$$\mathrm{Mod}(\mu_1) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{b,c\}, \{a,b,c\}\}$$

and

$$\mathrm{Mod}(\mu_2) = \{\emptyset, \{b\}, \{c\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

Thus,

$$\mathrm{Mod}(\neg(\mu_1 \wedge \mu_2)) = \{\{a\}, \{a,b\}, \{a,c\}\}.$$

Assume that we have the following order on the interpreations $\{a,b\} < \{a,c\}$.

On the one hand, $\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. This set is closed under $\wedge$ and thus $\mathrm{Mod}(\psi -^{p_\wedge} (\mu_1 \wedge \mu_2)) = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}\}$. On the other hand $\mathcal{M} = \mathrm{Mod}(\psi - \mu_1) = \{\emptyset, \{a,b\}, \{a,c\}, \{a,b,c\}\}$ is not closed under $\wedge$. Thus,

$$\begin{aligned} \mathrm{Mod}(\psi -^{p_\wedge} \mu_1) &= p_\wedge(\mathrm{Mod}(\psi - \mu_1)) \\ &= Cl_\beta(\mathrm{Mod}(\psi) \cup Min_\leq(\mathcal{M} \cap \mathrm{Mod}(\neg\mu_1))) \\ &= Cl_\beta(\{\emptyset, \{a,b,c\}\} \cup Min_\leq(\{\{a,b\},\{a,c\}\})) \\ &= \{\emptyset, \{a,b\}, \{a,b,c\}\} \text{ for } \{a,b\} < \{a,c\}. \end{aligned}$$

Moreover $\mathrm{Mod}(\psi - \mu_2) = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\}$, which is closed under $\wedge$. Thus, $\mathrm{Mod}(\psi -^{p_\wedge} \mu_2) = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\}$. Note that $\{a,c\} \in \mathrm{Mod}(\psi -^{p_\wedge} (\mu_1 \wedge \mu_2))$ and $\{a,c\} \notin \mathrm{Mod}(\psi -^{p_\wedge} \mu_1) \cup \mathrm{Mod}(\psi -^{p_\wedge} \mu_2)$, that is to say $\psi -^{p_\wedge} (\mu_1 \wedge \mu_2) \not\models \psi -^{p_\wedge} \mu_1 \vee \psi -^{p_\wedge} \mu_2$. This proves that $-^{p_\wedge}$ violates (C6) in $\mathcal{L}_{Horn}$.

Finally, for $\mathcal{L}_{Krom}$ formulas ψ, μ_1 and μ_2 in $\mathcal{L}_{Krom}$ having as sets of models $\mathrm{Mod}(\psi) = \{\{a,b,c,d\}\}$,

$$\begin{aligned} \mathrm{Mod}(\mu_1) = \; & \{\{a,c\},\{b,d\},\{a,b\},\{c,d\},\{a,b,c\},\{a,c,d\}, \\ & \{a,b,d\},\{b,c,d\},\{a,b,c,d\}\} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Mod}(\mu_2) = \; & \{\{a,b\},\{c,d\},\{a,d\},\{b,c\},\{a,b,c\},\{a,c,d\}, \\ & \{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}. \end{aligned}$$

can be used to prove that $-^{Cl_{\mathrm{maj}_3}}$ violates (C6) in $\mathcal{L}_{Krom}$.

□

For postulate (C7), we get a positive and a negative result.

Proposition 11. *Let $-$ be a contraction operator and $\mathcal{L}'$ a β-fragment. If $-$ satisfies postulate* (C7)*, then so does the refined operator $-^{Cl_\beta}$ in $\mathcal{L}'$.*

Proof. (C7) states that if $\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2) \not\models \mu_1$ then $\psi -^{Cl_\beta} \mu_1 \models \psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)$. Assume thar $\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2) \not\models \mu_1$, i.e. $Cl_\beta(\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2))) \nsubseteq \mathrm{Mod}(\mu_1)$. Since $\mu_1 \in \mathcal{L}'$, $Cl_\beta(\mathrm{Mod}(\mu_1)) = \mathrm{Mod}(\mu_1)$. We have $Cl_\beta(\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2))) \nsubseteq Cl_\beta(\mathrm{Mod}(\mu_1))$. By monotonocity of the closure operator it follows that $\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2)) \nsubseteq \mathrm{Mod}(\mu_1)$. Since $-$ satisfies (C7), we have $\mathrm{Mod}(\psi - \mu_1) \subseteq \mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2))$. By monotonocity of the closure operator it follows that $Cl_\beta(\mathrm{Mod}(\psi - \mu_1)) \subseteq Cl_\beta(\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2)))$. Hence, $\mathrm{Mod}(\psi -^{Cl_\beta} \mu_1) \subseteq \mathrm{Mod}(\psi -^{Cl_\beta} (\mu_1 \wedge \mu_2))$, thus proving that $\psi -^{Cl_\beta} \mu_1 \models \psi -^{Cl_\beta} (\mu_1 \wedge \mu_2)$. □

Proposition 12. *Let $- \in \{-_D, -_S\}$ and $\mathcal{L}' \in \{\mathcal{L}_{Horn}, \mathcal{L}_{Krom}\}$. Then, the refined operators $-^{p_\beta}$ violates postulate* (C7) *in $\mathcal{L}'$.*

Proof. Let $- \in \{-_D, -_S\}$.

Let us first consider $\mathcal{L}' = \mathcal{L}_{Horn}$. Let ψ, μ_1 and μ_2 be Horn formulas having as sets of models

$$\mathrm{Mod}(\psi) = \{\{a,b\}\},$$

$$\begin{aligned} \mathrm{Mod}(\mu_1) = \; & \{\emptyset,\{a\},\{b\},\{c\},\{d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\}, \\ & \{b,d\},\{c,d\},\{a,c,d\},\{b,c,d\},\{a,b,c,d\}\} \end{aligned}$$

and

$$\mathrm{Mod}(\mu_2) = \{\emptyset, \{c\}, \{d\}, \{a,b\}, \{c,d\}, \{a,b,c,d\}\}.$$

We have

$$\begin{aligned} \mathrm{Mod}(\neg(\mu_1 \wedge \mu_2)) = \; & \{\{a\},\{b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\}, \\ & \{a,b,c\},\{a,b,d\},\{b,c,d\},\{a,c,d\}\}. \end{aligned}$$

Assume that we have the following order on interpretations $\{a,b,c\} < \{a,b,d\} < \{a\} < \{b\}$.

On the one hand we get

$$\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a,b\},\{a\},\{b\},\{a,b,c\},\{a,b,d\}\}.$$

This set is not closed under $\wedge$. According to the order on interpretations $\mathrm{Mod}(\psi -^{p_\wedge} (\mu_1 \wedge \mu_2)) = \{\{a,b\},\{a,b,c\}\} \nsubseteq \mathrm{Mod}(\mu_1)$. On the other hand $\mathrm{Mod}(\psi - \mu_1) = \{\{a,b\},\{a,b,c\},\{a,b,d\}\}$, which is closed under $\wedge$. Therefore, $\mathrm{Mod}(\psi -^{p_\wedge} \mu_1) = \{\{a,b\},\{a,b,c\},\{a,b,d\}\}$. Note that $\mathrm{Mod}(\psi -^{p_\wedge} \mu_1) \nsubseteq \mathrm{Mod}(\psi -^{p_\wedge} (\mu_1 \wedge \mu_2))$, which means that $\psi -^{p_\wedge} \mu_1 \not\models \psi -^{p_\wedge} (\mu_1 \wedge \mu_2))$, thus proving that $-^{p_\wedge}$ violates le postulat (C7) in $\mathcal{L}_{Horn}$.

Let us now turn to the Krom fragment. Consider two Krom formulas, ψ and μ_1, having as sets of models

$$\mathrm{Mod}(\psi) = \{\{a,b,c,d\}\}$$

and

$$\begin{aligned} \mathrm{Mod}(\mu_1) = \; & \{\{a,c\},\{b,d\},\{a,b\},\{c,d\},\{a,b,c\},\{a,c,d\}, \\ & \{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}. \end{aligned}$$

Let μ_2 be the formula obtained from μ_1 in exchanging the roles of c and d:

$$\begin{aligned} \mathrm{Mod}(\mu_2) = \; & \{\{a,b\},\{c,d\},\{a,d\},\{b,c\},\{a,b,c\},\{a,c,d\}, \\ & \{a,b,d\},\{b,c,d\},\{a,b,c,d\}\}. \end{aligned}$$

Assume that we have the following order on interpretations $\{a,d\} < \{b,c\} < \{a,c\} < \{b,d\}$.

On the one hand,

$$\mathrm{Mod}(\psi - (\mu_1 \wedge \mu_2)) = \{\{a,b,c,d\},\{a,c\},\{a,d\},\{b,c\},\{b,d\}\},$$

which is not closed under maj_3 (e.g. $\{a,c,d\}$ is missing). According to the order on interpretations, $\mathrm{Mod}(\psi -^{p_{\mathrm{maj}_3}} (\mu_1 \wedge \mu_2)) = \{\{a,b,c,d\},\{a,d\}\} \nsubseteq \mathrm{Mod}(\mu_1)$. On the other hand $\mathrm{Mod}(\psi - \mu_1) = \{\{a,b,c,d\},\{a,d\},\{b,c\}\}$, which is closed under maj_3. Therefore $\mathrm{Mod}(\psi -^{p_{\mathrm{maj}_3}} \mu_1) = \{\{a,b,c,d\},\{a,d\},\{b,c\}\}$. Note that $\mathrm{Mod}(\psi -^{p_{\mathrm{maj}_3}} \mu_1) \nsubseteq \mathrm{Mod}(\psi -^{p_{\mathrm{maj}_3}} (\mu_1 \wedge \mu_2))$, which means $\psi -^{p_{\mathrm{maj}_3}} \mu_1 \not\models \psi -^{p_{\mathrm{maj}_3}} (\mu_1 \wedge \mu_2))$, thus proving that $-^{p_{\mathrm{maj}_3}}$ violates (C7) in $\mathcal{L}_{Krom}$. □

5 CONCLUDING DISCUSSION

We have investigated to which extent established model-based belief change operators can be refined to work within propositional fragments. We have first defined desired properties any refined belief change operator should satisfy and provided a characterization of all such refined operators. We have then focused on contraction. Our study was carried out in the context of model-based contraction initiated by Katsuno and Mendelzon [18] and enriched by Caridroit et al. [3]. It contributes to popularize this approach, which allows one to study contraction operators in propositional fragments from the models point of view within a suitable formal framework.

Compared to revision and update, refining contraction operators is more involved. In order to obtain rational contraction operators the notion of refinement has to be specified. It requires to take into account not only the result of the initial contraction, but also two additional parameters, the initial belief set and the information to be removed. We have provided concrete refined contraction operators. We have shown that they satisfy the basic postulates, whereas the recovery postulate $(\mathrm{C4})$ and the postulates dealing with the minimality of change $(\mathrm{C6})$ and $(\mathrm{C7})$ are more problematic.

In contrast to previous work on belief contraction that was mainly devoted to the Horn logic, our approach applies to any propositional fragment captured via closure properties on sets of models.

In the Horn case the proposed refined contraction operators provide new operators, that can be compared to two families of model-based contraction operators previously proposed within the Horn fragment, namely *Model-based Horn Contraction* (MHC) [25] and *Maxi Choice Horn Contraction based on Weak Remainder Sets* (MCHCWR) [8].

The closure-based refinement coincides with MCH in the special case where the initial contraction operator is defined by $\psi - \mu = \mathrm{Mod}(\psi) \cup Min(\mathrm{Mod}(\neg\mu), \leq_\psi)$ where $\leq_\psi$ is a faithful preorder over interpretations. This is the case, in particular, for Dalal's and Satoh's contraction operators. Note that, more generally, for any contraction operator satisfying (C1), (C2), (C3), (C5) and (C7), the closure-based refinement provides a contraction operator which operates within the Horn fragment and which satisfies these postulates as well.

The p_β-refinement can behave on some instances as an MCHCWR operator (but is not such an operator). Indeed, when the result of the initial contraction is not closed, then $\mathrm{Mod}(\psi -^{p_\beta} \mu) = Cl_\beta(\mathrm{Mod}(\psi) \cup \{m\})$ where $m \in \mathrm{Mod}(\neg\mu)$. However, while for an MCHCWR operator the choice of $m \in \mathrm{Mod}(\neg\mu)$ is arbitrary, in the case of p_β-refinement this model has to be chosen in $\mathrm{Mod}(\psi - \mu) \cap \mathrm{Mod}(\neg\mu)$. As such it corresponds to an instantiation of an MCHCWR operator which obeys to the principle of minimal change. Let us examine once more Example 1. No matter what is the fixed order on the interpretations, the model $\{a, b, c\}$ (which is a counter-model of μ and as such a valid candidate for an MCHCWR operator) will never be considered as a candidate to be in the result of the contraction by our refined operator. Indeed it is further away from ψ than any other counter-model of μ (e.g. for Dalal's contraction operator, for any model $m \in \mathrm{Mod}(\psi -_D \mu) \cap \mathrm{Mod}(\neg\mu)$, $min\{|m'\Delta m| : m' \in \mathrm{Mod}(\psi)\} = 1$, while $min\{|m'\Delta\{a, b, c\}| : m' \in \mathrm{Mod}(\mu)\} = 2$).

Natural extensions of this work are to study contraction when only the formula representing the belief set is in the fragment but not the formula representing the information to be removed, or when only the formula representing the information to be removed but not the formula representing the belief set. Our approach can handle these extensions.

We plan to continue our study in exploring systematically other belief change operations, in particular belief erasure, which is to contraction as update is to revision.

Besides, more ambitious issues could be investigated, namely the computational complexity of refined contraction operators, and from another point of view, the existence of decomposable characterizable fragments, which would give more general results on the satisfaction or not of postulate $(\mathrm{C4})$.

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