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Granular Mereogeometry

Hedda Rahel SCHMIDTKE [1]
University of Oregon, USA

Abstract. This paper presents a new mereological approach to formalizing geometric notions of incidence, congruence, and parallelism over extended regions. The axiomatization was built extending a decidable pre-mereological base language, showing where the geometric framework requires first-order extension. Important outcomes are in the investigation of how to define incidence between extended geometric objects that are suitable candidates to replace points, lines, and planes in a purely region-based first-order framework. Moreover, the mereogeometric approach proposed is shown to have a key advantage in allowing dimensionality to be a relative concept in contrast to it being an absolute concept encoded in the types of geometric entities. Especially this property, one may conclude, could make mereogeometry attractive for formalizing geometric relations in a cognitively adequate manner for applications that require the same flexibility of switching between conceptualizations of space of different dimensionality as human beings show in language use.

Keywords. mereogeometry, granularity, ontology, reasoning

1. Introduction

Since Euclid's times, the fundamental entities of geometry are extensionless entities tied to a dimension. Euclid's and later Hilbert's axioms start from 0-dimensional points, describe their relation to 1-dimensional lines, and 2-dimensional planes, with the dimensionality of space fixed to three by axioms over these lower-dimensional entities. A key concept forming the core of the classical axiomatizations [1,2] is incidence: points can lie on (*incide with*) a line or plane and lines a plane. Incidence, apart from ranging over extensionless entities instead of regions, seems intuitively very close to a part-of relation, so it seems intuitively natural to translate incidence into part-of. However, when we say "this point lies on that line," and translate this accordingly what we may need to claim is: this (somehow) point-like region, whose extensions do not matter, is part of a line-like region, which has only one relevant extension. The at first seemingly innocent effort in translation leads to existence claims about relevant or irrelevant extensions.

The lack of representation of extension in information systems is not only a theoretical concern, but a frequent reason of usability issues causing causing harm to people as well as economic damage. An IP address look-up service, for instance, may display a marker on a map at a precise, arbitrarily zoomable coordinate location *meant* to indicate the center of mass of a state to show that the IP address from which a fraud originated

[1]Corresponding Author: H. R. Schmidtke, 1251 University of Oregon, Eugene OR 97403-1251, USA; E-mail: schmidtke@acm.org.

is tied to that state. While the marker is not *meant* to indicate the particular building at that position, it may be mistakenly assumed to do so. Similarly, considerable damage is caused each year by GPS navigation systems that represent an extended location by a point, thus, e.g., indicating to a driver that they are still 500 m away from a lake while they are, in fact, 500 m away from the center of mass of the lake.

The information systems in these cases could have provided information about granularity. While it may be infeasible, unnecessary, or even undesirable to store and display information about the boundary of an extended region in these cases, the level of granularity that is *intended* could help users to better understand the information. We need to know at least the rough size of the entity indicated by a marker: does it indicate a 10 m radius or a 1 km radius radius? From the point of view of representational economy, a 2D or 3D coordinate does not require relevantly more space than a 2D+1 or 3D+1 coordinate containing granularity. Moreover such a representation will function even in regions with contested boundaries, where deciding for one alternative is undesirable and may cause diplomatic or economic inconvenience to a location provider.

Granular mereogeometry is a tool with which conceptual discrepancies between user and system can be uncovered. Mereogeometry, a theory of space and spatial relations aiming to formalize geometric notions within a region-based, mereological framework is an attractive alternative to geometry based on points and point-sets, in the same manner that mereotopology is an attractive alternative to topology based on point-sets. Mereogeometry was first proposed by Tarski [3]. However, while mereotopology has become a widely used framework employed within a wide range of domain ontologies, mereogeometry – after initial setbacks showing that certain characterizations following the Tarskian strategy are undecidable – received less attention and is not employed in any of the widely cited ontological frameworks. These negative results notwithstanding, the recent modern reconstruction [4] using a language similar to Protothetic, the logical language devised by Leśniewski [5], within which Tarski developed the first mereogeometry has brought renewed interest. The formal framework presented by [4] brings Protothetic and Tarski's mereogeometry into the format of the theorem prover Coq [6], thus making the original development more easily accessible and amenable to study.

2. Contributions

This paper aims to take a step towards developing a decidable, versatile mereogeometric framework. Based on a simple pre-mereological base language [7], axiom groups for incidence, congruence, and parallels are described and related to Hilbert's well-known axiomatization. While not reaching the long-term objective of providing a decidable mereogeometric framework, the axiomatization in this paper produces several results:

1. **Relative dimension.** Dimensionality in the proposed theory is characterized not through types but based on mereological properties, such as that no straight line is part of a point and no plane is part of a line. Incidence is characterized as the relation separating dimensions, it holds between a geometric entity of lower dimensionality and a geometric entity of higher dimensionality. Relative dimension also allows avoiding repetition of axioms for all properties for which, e.g., lines behave with respect to planes as points behave with respect to lines. It also allows leaving the maximal number of dimensions open.

2. **Unigranular entities.** All extensionless entities of classical geometry, such as points, lines, and planes are replaced by unigranular entities that are point-like, line-like, etc. entities with a specific granularity. Granularity of unigranular entities is used to replace congruence: all entities in the geometry are regions and have a granularity, i.e., spheres replace points, but straight lines and planes also have a thickness. Incidence relates entities of the same granularity: spheres of a certain diameter are incident with extended straight lines of the same diameter.

Besides the axiom group of continuity, which requires second-order formalizations, two axiom groups are missing: axioms for ordering and axioms for angle congruence.

Structure of the Article The next section provides a brief overview over the mereogeometric work program and outlines the main differences between this paper and related works. Section 4 introduces the logical language presenting its basic reasoning capabilities and the necessary extensions required to apply it to the formalization of mereogeometry. The main sections Sect. 5 and Sect. 6 introduce the pre-mereological base language and the main geometric characterizations built upon it.

3. Related Works

The main advantage of using mereology over a set-theoretic approach starting from points, whether in mereotopology or in mereogeometry, is the more natural characterization of spatial relations in terms of regions, which can be thought of as positions occupied by physical objects: in mereotopology and mereogeometry, points are an abstract, derived notion and the mereological framework makes this obvious. Consequently, mereotopology has proven to be a powerful tool to provide detailed models of topological properties in foundational ontologies [8] and application ontologies [9], and a range of calculi and spatial inference methods based on mereotopology exist [10]. When it comes to geometric properties, however, the most widely used calculi such as [11] are point-based and not compatible with a mereotopological view.

The first to propose a mereogeometry was Tarski [3]. Like in Tarski's approach, the approach proposed here starts from spheres. Tarski then proceeds to define points from spheres using the filter method. This facilitates the characterization of geometry and allowed Tarski to show that it is possible to build a geometry upon a purely mereological basis. However, introducing points as infinitesimal entities requires sacrificing decidability and current approaches following the Tarskian method, such as [12,13], confirmed this. Accordingly, mereogeometry, although an attractive variant for characterizing spaces in ontologies is generally not used in domain ontologies for characterizing spatial aspects. Dapoigny and Barlatier [4], for instance, show how application is possible within an interactive proof system able to handle second-order logics.

The idea of relating granularity to the extension of spheres in a Tarskian mereogeometry was proposed by Polkowski within the framework of rough mereology [14]. Like Tarski, however, he bases the notion of congruence not on the extension of spheres but on the equidistance of points.

While a discussion of the philosophical advantages of mereogeometry is beyond the scope of this paper and would in large parts repeat what has been said in justifying the need for mereotopology, I would like to draw attention to the ontological advantages a granular mereogeometry has. A fundamental phenomenon is the geometric nature of light: light seemingly travels on straight lines. In fact, space is characterized in modern physics in terms of these possible paths. The notion of a straight path is also crucial for other sciences, such as ecology: whether the wolfs see the deer in the forest depends on whether the straight path between them is sufficiently obstructed. A spatial notion of ecology such as *habitat* [15] or *forest* [16] is difficult to formalize in terms of mereotopology alone but easily characterized as depending on geometric notions of whether a certain region is *reachable* by enemies or natural forces, such as visible light or wind [17]. Not all of these phenomena travel on the straight paths of a Euclidean geometry but notions of directness and parallelism are exemplified by many phenomena, whose formal specification accordingly would benefit from geometric notions.

The notion of granularity is likewise intuitive under this perspective [17]. The granularity of a sand storm is the size of its *grains*. A point in a geometry for sandstorms should have the size of a grain; its straight lines with a given set of possibly changing directions are the possible straight paths of grains. A house that is properly sealed against sandstorms does not offer a straight line path having a thickness of the size of a sand grain. These levels of granularity are fundamental for specifying what is a shielded region with respect to a certain phenomenon: a board partition with centimeter-wide gaps can protect against wolfes, a clay hut against sand storms, and it takes the small bond length and atomic radius of lead to provide shielding from gamma rays.

In contrast to previous work on granular geometries [17,18], the aim in this paper is to present first steps towards a truly region-based granular geometry that avoids any introduction of spatial entities that lack extension. Furthermore, the limitations in [17, 18] to two-dimensional spaces is lifted. The granular mereogeometry presented here is flexible both with regard to whether a minimal granularity (i.e. atoms in a philosophical sense) exists and how many dimensions a space has.

4. Logical Language

The theory is formalized in a logical language derived from context logic [7]. Context logic is a logical language based on the usual propositional logical connectives together with a single binary operator, the pre-mereological $\sqsubseteq$ relation. For characterizing the geometry, the system of [7] is extended with a mechanism that allows to reflect universal and existential quantification.

4.1. Syntax of Context Logic

The syntax of context logic, as introduced in [7] and extended in [19], features two layers of logical descriptions: on the one hand, *context terms* represent contexts and allow the construction of contexts from other contexts using a sum operator $\sqcup$, an intersection operator $\sqcap$, and a complement $-$; on the other hand, formulae can be constructed using any of the propositional connectives. The pre-mereological $\sqsubseteq$, representing the relation between two contexts, is the only relation built into the language. Other relations, such as *spatial part of* or *subclass* can be developed from it, by interpreting $\sqsubseteq$ in a context. The key idea is to identify contexts with portions of reality. By removing the distinctions between individuals, types, predicates, and functions, on a basic level, the language thus realizes a pre-mereological basis in which extended portions of a domain are composed of extended portions of the domain. The only basic type of entities in context logic are contexts. Context terms are defined over an alphabet V of context variables:

1. For any $c \in V$, c is a context term.
2. The two constants $\top$ and $\bot$, called the universal and the empty context, are context terms.[2]
3. For any context term c, $-c$ is a context term.
4. For any two context terms c and d, $(c \sqcap d)$ and $(c \sqcup d)$ are context terms.

Formulae are constructed out of context terms using $\sqsubseteq$ as the only operator. Formulae are constructed out of other formulae using propositional logic connectives.[3]

1. For any three context terms c, d and e, $[c \sqsubseteq d]$ and $e : [c \sqsubseteq d]$ are formulae, called atomic formulae. The formula $e : [c \sqsubseteq d]$ is called a contextualized atom.
2. For any formula ϕ, $\neg\phi$ is a formula.
3. For any two formulae ϕ and ψ, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \rightarrow \psi)$ are formulae.

Contextualized atoms allow us to characterize mereological relations such as *spatial part of* as $\sqsubseteq$ in a spatial context.

4.2. Inference Schemata

The valid inference schemata are described in terms of a tableau reasoning procedure following [7]. A tableau reasoner expands pairs of sets of formulae called tableaux $t = \langle \Gamma, \Delta \rangle$ until either no further expansion is possible, in which case the tableaux are called *saturated*, or until a contradiction is found, which is the case if the same formula appears in both the Γ set and the Δ set of a tableau. In the former case, the algorithm answers "satisfiable" and the model can be read from the tableaux; in the latter case, the algorithm answers "unsatisfiable." Propositional logic inference is captured by the following rules:

$$\phi \wedge \psi \in \Gamma \vdash \phi \in \Gamma \text{ and } \psi \in \Gamma \qquad \text{(R1)}$$

$$\phi \wedge \psi \in \Delta \vdash \phi \in \Delta \text{ or } \psi \in \Delta \qquad \text{(R2)}$$

$$\phi \vee \psi \in \Gamma \vdash \phi \in \Gamma \text{ or } \psi \in \Gamma \qquad \text{(R3)}$$

$$\phi \vee \psi \in \Delta \vdash \phi \in \Delta \text{ and } \psi \in \Delta \qquad \text{(R4)}$$

$$\phi \rightarrow \psi \in \Gamma \vdash \phi \in \Delta \text{ or } \psi \in \Gamma \qquad \text{(R5)}$$

$$\phi \rightarrow \psi \in \Delta \vdash \phi \in \Gamma \text{ and } \psi \in \Delta \qquad \text{(R6)}$$

$$\neg\phi \in \Gamma \vdash \phi \in \Delta \qquad \text{(R7)}$$

$$\neg\phi \in \Delta \vdash \phi \in \Gamma \qquad \text{(R8)}$$

Further inference rules regard the handling of context terms. Here c, d, and e stand for arbitrary context terms.

$$[c \sqsubseteq d] \in \Gamma \vdash [\top \sqsubseteq c] \in \Delta \text{ or } [\top \sqsubseteq d] \in \Gamma \qquad \text{(R9)}$$

$$[\top \sqsubseteq -c] \in \Gamma \vdash [c \sqsubseteq \bot] \in \Gamma \qquad \text{(R10)}$$

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[2] Note that the inclusion of the empty context $\bot$ does not require us to assume the existence of an empty context according to the semantics laid out in [7]. It is, however, a powerful construction to express an impossible context. Using this symbol we can represent non-existence of a context and, for instance, tautology $[\bot \sqsubseteq \top]$ and contradiction $[\top \sqsubseteq \bot]$.

[3] We apply bracket saving rules to facilitate brevity and readability of expressions: outer brackets can be omitted, and connectives have the following order of precedence: $-, \sqcap, \sqcup, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$, where $(\phi \leftrightarrow \psi)$ is an abbreviation for $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$.

$$[\top \sqsubseteq -c] \in \Delta \vdash [c \sqsubseteq \bot] \in \Delta \tag{R11}$$

$$[\top \sqsubseteq c \sqcap d] \in \Gamma \vdash [\top \sqsubseteq c] \in \Gamma \text{ and } [\top \sqsubseteq d] \in \Gamma \tag{R12}$$

$$[\top \sqsubseteq c \sqcap d] \in \Delta \vdash [\top \sqsubseteq c] \in \Delta \text{ or } [\top \sqsubseteq d] \in \Delta \tag{R13}$$

$$[\top \sqsubseteq c \sqcup d] \in \Gamma \vdash [c \sqsubseteq \bot] \in \Delta \text{ or } [d \sqsubseteq \bot] \in \Delta \tag{R14}$$

$$e : [c \sqsubseteq d] \in \Gamma \vdash [e \sqcap c \sqsubseteq d] \in \Gamma \tag{R15}$$

$$e : [c \sqsubseteq d] \in \Delta \vdash [e \sqcap c \sqsubseteq d] \in \Delta \tag{R16}$$

In comparison to [7], there are additional rules for contextualized atoms (R15) and (R16), which will be inspected more closely below.

We see that there are not yet rules for handling the situations $[c \sqsubseteq d] \in \Delta$ and $[\top \sqsubseteq c \sqcup d] \in \Delta$. Rules for these cases can be defined on the basis of Hintikka systems. A Hintikka system in context logics is a pair $H = \langle T, S \rangle$ where S is a binary ordering relation over tableaux, i.e. reflexive, transitive, and antisymmetric, and T is a non-empty set of disjoint, saturated tableaux. We can now describe three additional rules.

For any tableaux $t = \langle \Gamma, \Delta \rangle$ and $t' = \langle \Gamma', \Delta' \rangle$, with $t, t' \in T$ and $S(t, t')$:

$$\text{for all atomic formulae } \phi : \phi \in \Gamma \text{ implies } \phi \in \Gamma' \tag{R17}$$

For any tableau $t = \langle \Gamma, \Delta \rangle$ with $[c \sqsubseteq d] \in \Delta$, there is a tableau $t' = \langle \Gamma', \Delta' \rangle$, with $t, t' \in T$ and $S(t, t')$:

$$[\top \sqsubseteq c] \in \Gamma' \text{ and } [\top \sqsubseteq d] \in \Delta' \tag{R18}$$

For any tableau $t = \langle \Gamma, \Delta \rangle$ with $[\top \sqsubseteq c \sqcup d] \in \Delta$, there is a tableau $t' = \langle \Gamma', \Delta' \rangle$, with $t, t' \in T$ and $S(t, t')$:

$$[c \sqsubseteq \bot] \in \Gamma' \text{ and } [d \sqsubseteq \bot] \in \Gamma' \tag{R19}$$

A sentence κ follows from a knowledge base KB iff the tableau $t = \langle KB, \{\kappa\} \rangle$ is unsatisfiable. To prove this we use the fact that a Hintikka system for t exists if and only if $KB \not\models \kappa$. The above rules give rise to a non-deterministic algorithm, which aims to construct a Hintikka system $\langle T, S \rangle$ for t starting from $T_0 = \{t\}$ and $S_0 = \{\langle t, t \rangle\}$. Intuitively, the tableaux in T describe the properties that hold within contexts and the relation S captures information about the relation ($\sqsubseteq$) between contexts.

[7] showed that the resulting tableau method is sound with respect to the semantics defined there and weakly complete.[4] This result carries over, since the only additions we make are the rules (R15) and (R16), which eliminate the contextualization of atoms. Since there is no rule introducing these abbreviations, termination is not affected.

[4]The weak completeness stems from the fact that the algorithm does not expand formulae. Completeness in the strong sense would require that the algorithm derive all consequences of a formula.

4.3. Additional Language Constructs

For characterizing the geometric framework, further constructions are required, which bring the language to the expressiveness of first-order logic.

The definition arrow $\overset{def}{\Leftrightarrow}$ signals that the construction on the left hand side should be replaced by the construction on the right hand side. The universal quantifier $\forall$ indicates that a replacement operation can be performed for all quantified contexts, i.e. as with a unification mechanism. The existential quantifier $\exists$ signals that a new, previously not used name for a context term should be introduced for the existentially quantified variable with variable entries for all universally quantified variables in whose scope it is, in the manner of Skolemization. While a set of formulae featuring only universally quantified formulae increases the complexity of the reasoning, as it abbreviates a set of expressions, the extension with both existentially and universally quantified formulae, when embedded into the reasoning procedure using a unification mechanism will reduce reasoning to semi-decidability. Geometry features many such axioms, e.g. that any line has at least two points.[5]

We introduce relations *equals* ($=$) (D1) and *overlaps* ($\bigcirc$) (D3) with contextual variants (D2) and (D4), respectively.

$$[c = d] \overset{def}{\Leftrightarrow} [c \sqsubseteq d] \wedge [d \sqsubseteq c] \qquad \text{(D1)}$$

$$[c \bigcirc d] \overset{def}{\Leftrightarrow} \neg[c \sqcap d \sqsubseteq \bot] \qquad \text{(D3)}$$

$$x : [c = d] \overset{def}{\Leftrightarrow} x : [c \sqsubseteq d] \wedge x : [d \sqsubseteq c] \qquad \text{(D2)}$$

$$x : [c \bigcirc d] \overset{def}{\Leftrightarrow} \neg x : [c \sqcap d \sqsubseteq \bot] \qquad \text{(D4)}$$

5. Mereological Foundation

The fundamental notion for building mereological relations is the *pre-mereological relation* $\sqsubseteq$. We can easily prove that $\sqsubseteq$ is a partial ordering relation. Using contextualized atoms, $\sqsubseteq$ can be used as a tool to specify mereological relations, such as *spatial part of* or *subclass of*. Any mereological relation x inherits these basic ordering properties:[6]

[5] In this paper, the scope of quantifiers is to be read as maximal, that is until a bracket closes that is opened before the quantifier or until the end of the formula.

[6] Since these notions are fundamental for the proposed theory, we sketch the proofs.

Proof: (1) follows directly from the definition of $=$ (D1). The atomic formula (2) corresponds to the fundamental tautology: $[\top \sqsubseteq c] \vee \neg[\top \sqsubseteq c]$. The formula (3) is expanded into the propositional formula: $([\top \sqsubseteq b] \vee \neg[\top \sqsubseteq a]) \wedge ([\top \sqsubseteq c] \vee \neg[\top \sqsubseteq b]) \rightarrow ([\top \sqsubseteq c] \vee \neg[\top \sqsubseteq a])$, which could only become false if $[\top \sqsubseteq c]$ were false and $[\top \sqsubseteq a]$ were true. For the antecedent to become true in this case, however, both $[\top \sqsubseteq b]$ and $\neg[\top \sqsubseteq b]$ would have to hold.

Equation (4) follows directly from the definition of $=$ in a context (D2). The atomic formula (5) is expanded into $[\top \sqsubseteq c] \vee \neg[\top \sqsubseteq c \sqcap x]$. If we assume $[\top \sqsubseteq c]$ and $\neg[\top \sqsubseteq c \sqcap x]$ are both false, then $[\top \sqsubseteq c]$ is false and $[\top \sqsubseteq c \sqcap x]$ is true. However, expanding the second formula using the Γ rule (R12) yields $[\top \sqsubseteq c]$ and $[\top \sqcap x]$, in contrast to the assumption that $[\top \sqsubseteq c]$ is false.

The formula (6) is expanded into the formula: $([\top \sqsubseteq b] \vee \neg[\top \sqsubseteq x \sqcap a]) \wedge ([\top \sqsubseteq c] \vee \neg[\top \sqsubseteq x \sqcap b]) \rightarrow ([\top \sqsubseteq c] \vee \neg[\top \sqsubseteq x \sqcap a])$. This could only become false if $[\top \sqsubseteq c]$ were false and $[\top \sqsubseteq x \sqcap a]$ were true. For the antecedent to become true now, both $[\top \sqsubseteq b]$ and $\neg[\top \sqsubseteq x \sqcap b]$ would have to hold. The latter holds if either $[\top \sqsubseteq x]$ or $[\top \sqsubseteq b]$ are false. Since $[\top \sqsubseteq b]$ is true, $[\top \sqsubseteq x]$ must be false, but this can also be ruled out, since $[\top \sqsubseteq x \sqcap a]$ and thus both $[\top \sqsubseteq x]$ and $[\top \sqsubseteq a]$ must be true.

$$[c \sqsubseteq d] \wedge [d \sqsubseteq c] \rightarrow [c = d] \quad (1) \qquad x : [c \sqsubseteq d] \wedge x : [d \sqsubseteq c] \rightarrow x : [c = d] \quad (4)$$

$$[c \sqsubseteq c] \quad (2) \qquad x : [c \sqsubseteq c] \quad (5)$$

$$[a \sqsubseteq b] \wedge [b \sqsubseteq c] \rightarrow [a \sqsubseteq c] \quad (3) \qquad x : [a \sqsubseteq b] \wedge x : [b \sqsubseteq c] \rightarrow x : [a \sqsubseteq c] \quad (6)$$

Spatial Mereology The foundation for the geometry is *spatial part of*. We characterize it using the context s, the formula $s : [a \sqsubseteq b]$ meaning that a is a spatial part of b. Intuitively, s focuses on the purely spatial aspects of a and b. Using the above results (4)-(6), we know that spatial mereology is an ordering relation.

Names and Classes Class hierarchies are represented with the context n, the formula $n : [a \sqsubseteq b]$ meaning that a is a subclass of b. To use a familiar example: if we want to express that apple trees belong to the rose family, we could write this as $n : [\mathit{apple\_trees} \sqsubseteq \mathit{rose\_family}]$. Class contexts, such as *apple_trees* would not need to have a spatial extension. However, an apple tree in the garden could have both a class name and a location:

$$n : [\mathit{apple\_tree14} \sqsubseteq \mathit{apple\_trees}] \wedge s : [\mathit{apple\_tree14} \sqsubseteq \mathit{garden123}]$$

We can think of the relation $n : [a \sqsubseteq b]$ as a relation between labels that can be attached to entities a and b. In this case, $n : [a \sqsubseteq b]$ would mean that $n \sqcap a$ contains all labels attached to b. We can also think of $n : [a \sqsubseteq b]$ as a relation between sets of unique IDs. In this case, $n : [a \sqsubseteq b]$ would mean that the set $n \sqcap a$ contains only unique IDs that are also in b.

This example illustrates that context terms can take the roles of relation names, e.g. the spatial part-of obtained from the context s, of classes, e.g. the class *rose_family*, and of individuals, e.g. the object *apple_tree14*. When thinking within the familiar set-theoretical frame, it may be startling to eliminate the separation between sets and their elements. However, it is a main advantage of mereology to avoid this distinction and with it the Russell antinomy. When translating a granular mereological framework into the familiar set-theoretical framework, it may therefore be helpful to picture individuals as singleton sets, to view granularity as the mechanism to distinguish cardinalities of classes, and, if desired, to single out individuals as atoms in the context n. For the purposes of this paper, we require only the class u of unigranular entities, our geometric base entities.

Linear Extensions In addition to the mereological basis, we assume linear ordering relations extending the mereological partial ordering relations. This notion will be fundamental for the mereogeometry, since several geometric notions require linear orderings. In particular, linearity is fundamental for the notions of dimension and granularity. A context d is a linear extension of a context c, iff all relations of c also hold in d and the relations in d are additionally linear.

$$\begin{aligned} \mathrm{linext}(d, c) \overset{def}{\Leftrightarrow} \; & (\forall x, y : c : [x \sqsubseteq y] \rightarrow d : [x \sqsubseteq y]) \wedge \\ & \forall x, y : [x \bigcirc c] \wedge [y \bigcirc c] \rightarrow d : [x \sqsubseteq y] \vee [y \sqsubseteq x] \end{aligned} \quad \text{(D5)}$$

The intuitive example of a linear extension is *spatial granularity*: if a is a spatial part of b, then a's granularity is smaller or equal to that of b. The linearity constraint ensures that arbitrary spatial entities can be compared making it possible to completely stratify the domain of spatial entities. The other important example is relevant dimensionality: any two spheres have the same dimension, but they are both of lesser dimension than a straight line or a slice. Note that the two notions are independent from each other: a straight line of small diameter is of higher dimension but smaller granularity than a sphere of larger diameter.

6. Geometric Axioms

With the base language defined, we can now proceed to characterizing the geometry. The task of axiomatizing geometry in context logic is the task of axiomatizing the context s and its linear extensions, as well as the context u characterizing the unigranular entities. This section starts with the characterization of granularity and dimensionality.

6.1. Granularity and Congruence

Aiming for an axiomatization built around not points, lines, and planes, but punctual, linear, and planar regions, granularity *gran* is defined as a linear extension of s referring to the extent, e.g. the diameter of a sphere or extended straight line. Granularity can also take the role that congruence between line segments has in classical geometry [18]. Congruence over unigranular entities relates spheres with a certain diameter to lines and planes having this diameter as their thickness. We characterize spatial extent (*gran*) as a linear extension over s (A1).

$$\text{linext}(gran, s) \tag{A1}$$

With the pre-mereological foundation, no other axioms are needed to specify the ordering *gran* establishes among entities of the same dimensionality. However, to specify the relation between entities of different dimensionality, we need to be able to relate to them.

6.2. Dimensionality

All axiomatizations from Euclid's [1] and Hilbert's [2] to Tarski's [3] and also the more recent variants of [4], [20], or [21] describe spaces of a fixed given dimensionality. In fact, axioms are written in such as way as to limit the dimensionality of space. However, it is remarkable that cognitive spaces such as exhibited in language use [22,23,24] seem to be flexible in such a way that human cognition is able to switch between conceptualized dimensionality. In fact, these simplifications are key for many cognitive spatial tasks human beings are able to perform, such as route planning and navigation.

This ability is closely tied to the notion of context. In one context, we represent an environment as a three-dimensional space in another context we choose a two-dimensional representation. Context logic has the advantage to allow this switch to be represented in a simple manner and the proposed geometry should also reflect this. Ignoring the actual dimensionality of a geometric object, we refer only to its dimensionality relative to other geometric objects: in several regards, a line behaves with respect to a plane, as a point behaves with respect to a line and a plane with respect to a three-dimensional space. The generally necessary constraints to obtain the common underlying relation between dimensions thus become visible. Moreover, we see that we may or may not posit minimal and maximal dimensions. Finally, the effort allows in several places to provide only a single axiom where the traditional formalization requires multiple.

We define dimensionality as a linear extension of spatial mereology (A2):

$$\text{linext}(dim, s) \tag{A2}$$

That *dim* is a linear extension of *s* means that if an entity is part of another then it has smaller or equal dimensionality. A sphere of Euclidean space, for instance, cannot contain a straight line, or planar slice. A slice can contain a slice, line, or sphere of smaller or equal granularity, but not a sphere of larger granularity. Dimensionality and its relation to granularity are further axiomatized with the notion of incidence. In fact, as incidence relates the types to each other in classical geometry, it fulfills this role with respect to dimensionality and granularity in the granular mereogeometry laid out here.

6.3. Mereogranular Axioms

Before presenting the characterization of classical geometric axioms within the proposed framework, fundamental notions such as incidence and parallels need to be connected to the mereogranular framework characterized so far.

Incidence Incidence is the crucial notion underlying many classical axiomatizations of geometry. Many variants of how to introduce incidence into the mereogeometry are possible and may lead to very different results. The basic idea in this paper is to characterize incidence (written with the Greek letter ι) as a relation derived from spatial part-of that holds only between unigranular entities of the same granularity. Moreover, the first parameter of $x \, \iota \, y$ should be of a specific lower dimensionality than the second.

We define incidence (D6) as holding between a lower-dimensional unigranular entity p and another unigranular entity x iff whenever p is a spatial part of the intersection between x and another entity y of the same dimensionality and granularity and another unigranular entity p' also fulfills these criteria, then p' is spatially part of p.[7] Intuitively, this definition characterizes the points (lines) incident with a line (plane) as the maximal unigranular entities contained in the intersections between the line (plane) and other lines (planes). As desired, it follows from the definition that incidence entails spatial $\sqsubseteq$, i.e.: a point on a line is a part of the line.

$$\begin{aligned} p \, \iota x \overset{def}{\Leftrightarrow} n : [p \sqsubseteq u] \wedge n : [x \sqsubseteq u] \wedge s : [p \sqsubseteq x] \wedge \forall y, p' : s : [p \sqsubseteq x \sqcap y] \wedge s : [p' \sqsubseteq x \sqcap y] \\ \wedge \, dim : [x = y] \wedge \neg s : [x = y] \wedge gran : [x = y] \wedge n : [p' \sqsubseteq u] \rightarrow \neg s : [p \sqsubseteq p'] \end{aligned} \tag{D6}$$

$$p \, \iota \, x \rightarrow gran : [p = x] \wedge dim : [p \sqsubseteq x] \wedge \neg dim : [x \sqsubseteq p] \tag{A3}$$

[7] Here and in the following, variable names are used in a manner to make the dimensional differences clear. p, p', p_1, for instance are employed to indicate a context of lower dimensionality, while t, t', t_1 are used for contexts of higher dimensionality. However, it should be noted that this procedure is for providing illustrative examples only. The language does not support types in this manner, the only way to express types of contexts is to relate them with respect to the class context n (see above). Dimensionality, however, is expressed with the context *dim*.

$$\forall p_1, p_2, t : p_1 \iota t \wedge p_2 \iota t \rightarrow gran : [p_1 = p_2] \tag{7}$$

Axiom A3 relates incidence to dimensionality and granularity: given any two entities, if one is incident with the other, they have the same granularity and the first has lower dimensionality than the second.

It follows directly from the transitivity of =, which follows from the transitivity of $\sqsubseteq$ that all extended points on a line have the same extension, that is, that the ordering between entities of higher dimensionality carries over in the same manner to entities of lower dimensionality (7).[8] Incidence thus establishes the stratification of the domain of unigranular entities according to dimensionality and granularity.

In contrast to classical axiomatizations, incidence as characterized here holds between unigranular entities of a certain dimensionality d and unigranular entities of the next higher dimensionality $d+1$, i.e. between points and lines, and lines and planes of the same granularity, but not between points and planes. We can derive the classical variant which ranges over several dimensions by first introducing n-level transitive extensions (D9) and then forming the transitive hull over several of these relations (D10). [9]

$$p \iota^2 x \overset{def}{\Leftrightarrow} \exists t : p \iota t \wedge t \iota x \tag{D7}$$

$$p \iota^{n+1} x \overset{def}{\Leftrightarrow} p \iota^n t \wedge t \iota x \tag{D9}$$

$$p \iota^3 x \overset{def}{\Leftrightarrow} p \iota^2 t \wedge t \iota x \tag{D8}$$

$$p \iota^* x \overset{def}{\Leftrightarrow} p \iota x \vee p \iota^2 x \tag{D10}$$

Parallels Before being able to present the classical axioms of incidence and parallelism, a further context needs to be introduced to be able to express a basic notion of directionality. We will not characterize ordering relations in this paper, but need one basic notion for specifying parallels as entities having the same direction. Two entities have the same direction if they either are spatially the same or they both lie on an entity x and there is no entity p that lies on both:

$$dir : [t_1 = t_2] \leftrightarrow s : [t_1 = t_2] \vee \exists x : t_1 \iota x \wedge t_2 \iota x \wedge \neg \exists p : p \iota t_1 \wedge p \iota t_2 \tag{A4}$$

It follows, that points or punctual entities, if they were introduced as entities with which no other entities are incident, trivially fulfill the requirement (A4).

6.4. Classical Axioms of Incidence and Parallelism

We can now proceed to characterize geometric properties in a way similar to classical axiomatizations to demonstrate the expressiveness of the proposed framework. Since these are widely known and well accessible, the characterizations follow the reference geometry of Hilbert [2]. These advantages notwithstanding, using the Hilbertian axioms is disadvantageous in so far as the here proposed axiomatization is more general. In order to prove several of the classical axioms, we need to make assumptions, e.g. about

[8] Note that this is not a contradiction to entities of lower dimensionality being strict parts of entities of higher dimensionality. The fact that, $s : [p \sqsubseteq x]$ holds but not $s : [x \sqsubseteq p]$ does not prevent the extension *gran* from adding $gran : [x \sqsubseteq p]$.

[9] We may remark that, although the $n+1$ index may suggest this, we do not lose decidability from these definitions, as we do not posit the existence of successor dimensions, from which an addition operator could be derived.

0-dimensional points, in order to limit generality. However, this does not demonstrate a limitation but rather the increased generality of the proposed granular mereogeometry.

Incidence Axioms Hilbert's axioms of incidence can be characterized now:

$$\forall p_1, p_2 : dim : [p_1 = p_2] \wedge gran : [p_1 = p_2] \wedge \neg s : [p_1 = p_2] \wedge \\ (dir : [p_1 = p_2] \vee \exists x : x \iota p_1 \wedge x \iota p_2) \\ \rightarrow \exists t : p_1 \iota t \wedge p_2 \iota t \tag{A5}$$

$$\forall p_1, p_2, t_1, t_2 : p_1 \iota t_1 \wedge p_2 \iota t_1 \wedge p_1 \iota t_2 \wedge p_2 \iota t_2 \rightarrow s : [t_1 = t_2] \vee s : [p_1 = p_2] \tag{A6}$$

$$\forall t : \exists p_1, p_2 : p_1 \iota t \wedge p_2 \iota t \wedge \neg s : [p_1 = p_2] \tag{A7}$$

$$\forall p_1, p_2, p_3 : dir : [p_1 = p_2] \wedge dir : [p_2 = p_3] \wedge dir : [p_1 = p_3] \\ \wedge \neg(\exists t : p_1 \iota t \wedge p_2 \iota t \wedge p_3 \iota t) \rightarrow \exists x : p_1 \iota^2 x \wedge p_2 \iota^2 x \wedge p_3 \iota^2 x \tag{8}$$

$$\forall p_1, p_2, p, t, x : p_1 \iota^2 x \wedge p_2 \iota^2 x \wedge p_1 \iota t \wedge p_2 \iota t \rightarrow t \iota x \tag{A8}$$

$$\forall x_1, x_2, p_1, p_2, p_3 : p_1 \iota^2 x_1 \wedge p_2 \iota^2 x_1 \wedge p_3 \iota^2 x_1 \wedge p_1 \iota^2 x_2 \wedge p_2 \iota^2 x_2 \wedge p_3 \iota^2 x_2 \\ \rightarrow s : [x_1 = x_2] \vee s : [p_1 = p_2] \vee s : [p_2 = p_3] \vee s : [p_1 = p_3] \tag{9}$$

$$\forall x_1, x_2, p_1 : p_1 \iota^2 x_1 \wedge p_1 \iota^2 x_2 \\ \rightarrow \exists p_2 : \neg s : [p_1 = p_2] \wedge p_2 \iota^2 x_1 \wedge p_2 \iota^2 x_2 \tag{A9}$$

$$\forall p, x, x' : p \iota^3 x \wedge p \iota^3 x' \rightarrow s : [x = x'] \tag{AD1}$$

$$\exists p, x : p \iota^3 x \tag{AD2}$$

The above variant of Hilbert's first axiom (A5) ensures that, if there are two distinct entities of a dimension with the same extent, which are parallel or share a lower-dimensional entity, then there is a higher dimensional entity with which they both incide. Axiom A6, corresponding to Hilbert's second axiom, describes that if entities p_1 and p_2 incide with entities t_1 and t_2, then either p_1 and p_2 or t_1 and t_2 are spatially the same. Hilbert's third axiom has two parts. The first part is reflected in (A7): for any entity t, there are at least two different entities that incide with it. The second part of Hilbert's third axiom ensures that there are at least three points that do not lie on the same line. We cannot require this, since the dimension neutral characterization would lead to this axiom entailing an unlimited number of dimensions.

Hilbert's fourth incidence axiom states that for any three points not on the same line there is a plane that contains them. We can derive this axiom in the form (8) by applying (A5) twice, when we assume that all points (trivially) have the same direction: from (A5) we know that there are lines t_{12} on which the points p_1 and p_2 lie and t_{23} on which the points p_2 and p_3 lie. Given t_{12} and t_{23} and knowing that they share the point p_2, we can again apply (A5) to derive the plane x. Note that (8) is a dimension independent theorem, that is we also derived that three parallel lines characterize a space

We state Hilbert's sixth incidence axiom (A8) before deriving his fifth (9) from it: the sixth axiom states that if an entity x contains in two steps entities p_1 and p_2 then x contains a complete entity t on which p_1 and p_2 lie. We can now derive a generalized version of the fifth incidence axiom, which states that for any three different points, there is no more than one plane on which they all lie. Assuming three different entities p_1, p_2 and p_3 lie on two entities x_1 and x_2 in two steps. From (A8), we know that t_{12} on which the points p_1 and p_2 lie and t_{23} on which the points p_2 and p_3 lie are incident on both x_1 and x_2. Applying (A6) and knowing that all of the p_i and thus t_{12} and t_{23} are distinct, we obtain that x_1 and x_2 must be equal.

Incidence axiom 7 (A9) states that two planes that have one point in common have at least one more point in common.

Incidence axiom 8 states that there are at least four points not lying on the same plane. This ensures that there is more than one plane. Following the goal to leave the dimensionality flexible so that, e.g., a reasoner that needs to solve a given problem could switch between representations, one may leave this property out. If a problem presents four points that are not in the same plane, the reasoner switches to the three-dimensional case, if not, a two-dimensional solution may be sufficient. We could ensure a minimum and a maximum for dimensionality by specifying the limit of ι-steps we allow or ensure: the example axioms (AD2) and (AD1), for instance, axiomatize an exactly three-dimensional geometry.

Axiom of Parallels Parallels and the context *dir* were introduced in (A4) and used in (A5) and (8). We can now characterize the axiom of parallels:

$$\begin{aligned} &\forall p,t : \neg p\,\iota\,t \wedge gran : [p = t] \\ &\quad \rightarrow \exists t' : p\,\iota\,t' \wedge dir[t = t'] \wedge \forall t'' : p\,\iota\,t'' \wedge dir[t = t''] \rightarrow s : [t'' = t'] \end{aligned} \tag{A10}$$

Axiom A4 characterizes parallels with respect to entities incident on them and entities on which they incide. Axiom A10 is the generalized version of Euclid's axiom of parallels.

7. Conclusions

This paper presented a new mereological approach to formalizing geometric notions of incidence, congruence, and parallelism. The axiomatization was built extending a decidable mereological base language, with first-order quantification.

Important outcomes are in the investigation of how to characterize incidence between granular geometric objects that are suitable candidates to replace points, lines, and planes in a purely region-based framework. Moreover, the mereogranular approach is shown to have a key advantage in allowing dimensionality to be a relative concept in contrast to it being an absolute concept encoded in the types of geometric entities. Especially this property, one may conclude, could make mereogeometry attractive for new applications and representations of common sense and cognitive concepts.

The paper leaves several open questions for future work. The long-term goal of a cognitively adequate and decidable mereogeometric framework will require further study both on the geometric side and on the reasoning side. Two important axiom groups are missing: axioms for ordering and angle congruence. Also, soundness with respect to set-theoretic models of geometry should be proven, which requires bridging the differences between the mereogeometric and the point-based algebraic language. On the side of the reasoner, nested existentially quantified axioms, such as the first incidence axiom which guarantees the existence of a line for any two points, can lead a reasoner to introduce considerable numbers of entities or even make it cease to terminate. These have to be handled carefully in a practically applicable mereogeometric reasoner.

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