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Mechanics and mathematics of rigid-plastic analysis — from the point of design methods —

Méchanique et mathématique de l'analyse plasto-rigide — du point de méthodes de design —

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ABSTRACT
Conventional manual-based design is being shifted to performance-based design in the geotechnical engineering practices. During these changes, it is not clear how to treat the conventional limit equilibrium methods for the bearing capacity, earth pressures and slope stability problems. Limit equilibrium methods are similar to the limit analysis. However, theoretical consistencies are lost in the limit equilibrium methods, because bold assumptions and empirical relations are included. Originally, rigid-plastic analysis can be very useful for the practical design because of its theoretical simplicity, fewer material parameters and its stability in the calculations. In this article, mechanics and mathematics of limit analysis are reviewed in comparison with other methods. Recent developments of numerical rigid-plastic methods are also reported. The author thinks these kinds of knowledge should be utilised for the reconstruction of plastic design procedures in geotechnical engineering practices.

RÉSUMÉ
En génie civil, les méthodes conventionnelles d'un design basé sur des normes et des manuels cédent peu à peu leur place á des méthodes basées sur les performances. Cependant, l'utilisation des méthodes conventionnelles d'équilibre limite pour résoudre les problèmes de capacité de chargement, de poussée des terres et de stabilité des pentes présente encore des zones d'ombres. Les méthodes d'équilibre limite sont similaires à l'analyse limite. Mais les fondements théoriques de ces méthodes manquent de rigueurs tant elles reposent sur des hypothéses trop audacieuses et nombre de relations empiriques. L'analyse plasto-rigide peut être trés utile pour la conception pratique grâce à sa simpliciteé théorique, à son nombre réduit de paramètres caractéristiques des matériaux et à sa stabilité numérique. Dans cet article, la méchanique et la mathématique de l'analyse limite sont comparées aux autres méthodes. Les récents développement de l'analyse plasto-rigide numérique sont aussi exposés. Lauteur pense que ces méthodes devraient être utilisées pour la mise au point de procédures de design plastique dans la pratique du génie civil.

1 INTRODUCTION

Recently, design methods are being shifted to performance-based design in order to establish a more rational design procedure. During these changes, it is inevitable to rearrange a framework of classical soil mechanics problems such as bearing capacity, earth pressure or slope stability problems. Although these classical formulations are still useful in many engineering practices, it seems to the author that patch works of the classical formulations for more general problems including soil reinforcement, complicated geometry or complicated strength distributions and so on, are almost impossible due to the theoretical ambiguities of the original formulation. It may be better to construct a new framework of the classical soil mechanics problems thoroughly based on the mechanics and mathematics of rigid-plastic analysis, because design methods are always required their accountability in a scientific manner. In addition to this, it is desirable to select a suitable and rational evaluation method for the required accuracies, importance of the structures, design costs or other technical constraints.

Generally, the following three kinds of errors are included when we solve initial-boundary value problems

(1) Errors due to the choice of physical and mathematical modelling.
(2) Errors due to the choice of parameters.
(3) Errors due to the choice of analytical methods.

The former two kinds of errors are well recognised. Choice of models deeply depends on the engineering judgement which is generally determined under the balances between designing costs and accuracies of the results. Similarly, choice of parameters mainly depends on the limited number of soils tests and field investigations. However, in comparison with the former two kinds of errors, we are likely to overlook the last errors. In case of rigid-plastic analysis, the errors due to analytical methods are investigated in this paper.

In practical design, limit equilibrium method is widely used due to its simplicity, for example. However, strictly speaking, mechanical and mathematical consistencies are lost in the formulation of limit equilibrium method. On the other hand, elastic-plastic finite element method (EPFEM)is sometimes used for the detailed design. In addition, results of EPFEM are sometimes directly compared with results of a conventional method. How should we understand the different results?

For another misleading example, in order to estimate the slope stability, stress fields obtained by elastic FEM are used for the calculation of the factor of safety along the assumed slip surfaces. This analysis may show a certain result. But what is the physical meaning of this analysis?

The author thinks that limit analysis (LA) plays a key role to understand the gaps of the results obtained by both limit equilibrium method (LEM) and EPFEM, as is briefly explained hereafter. Based on the limit theorems, we can derive a wide variety of formulations from the very simple manual calculations to precise rigid-plastic finite element method (RPFEM). LEM is somewhat similar to the very simple manual calculations of LA, because LEM is a certain kind of reduction of LA. In this sense, results by LEM sometimes seem similar to results by LA. But it should be noted again that the mechanical and mathematical consistencies are lost. Oppositely, by the use of spacial discretisation by finite elements, precise RPFEM can be formulated based on the limit theorems. Since RPFEM solves the governing equations directly, its results are theoretically correct except for the discretisation errors. The major difference of RPFEM and EPFEM is their material behaviour; simplified rigid-plastic constitutive relations are used in RPFEM.

Moreover, RPFEM can be applicable to general rigid-plastic boundary value problems including soil reinforcement, geometric complexity, complicated distribution of materials and so on, which are very difficult to be solved by the conventional limit equilibrium method. In this sense, As described above, it is understandable that limit analysis holds potentials for the updated approaches of the plastic design in geotechnical engineering. Fundamental features of limit analysis are briefly reviewed in the next section.

2 MATHEMATICAL STRUCTURE OF LIMIT ANALYSIS

2.1 *Features of limit analysis*

Most of the practical problems in geotechnical engineering essentially belong to elastic-plastic initial boundary value problems. Therefore, the strict way to solve these problems is elastic-plastic finite element method (EPFEM). To obtain the exact solutions, we should carry out the incremental form of EPFEM along with the precise construction procedure under suitable initial and boundary conditions. Generally speaking, these analyses are still hard because it is difficult to measure an initial stress field and it is also difficult to predict a detailed loading history of natural external loads. On the contrary, limit analysis (LA) based on the limit theorems is independent of the initial conditions and loading histories due to the simplification of material behaviour. Solutions obtained by LA are only the critical load factors and the detailed responses in time domain are no longer available. However, it is still useful for the practical engineering design for the following reasons.

- Obtained results by LA include a factor of safety or an ultimate load factor. These quantities are direct to the engineers' interest.
- Initial conditions and detailed loading histories are not necessary for LA.
- From the point of mathematics, LA belongs to convex programming. Therefore, solutions can converge to the globally optimum solution. Moreover, calculations of optimisation are stable.

2.2 *Limit theorems and their duality*

Limit theorems consist of two theorems, namely the lower bound theorem and the upper bound theorem.

The lower bound theorem insists that a system is plastically safe if there exists a certain stress field which satisfies the following two conditions;

- A statically admissible (SA) stress field which is satisfying the equilibrium of forces anywhere in the body and Neumann boundary conditions.
- A plastically admissible (PA) stress field which is not violating the yielding conditions anywhere in the body.

Load factors obtained by the lower bound theorem are always less than or equal to the exact solution.

On the other hand, the upper bound insists that a system is plastically stable if the internal dissipation rate is greater than the external plastic work rate for the following failure mechanism;

- A kinematically admissible (KA) velocity field which is satisfying the (plastic) strain rate $\sim$ velocity relationship anywhere in the body and Dirichlet boundary conditions.
- A plastically admissible (PA) velocity field which is satisfying the associated flow rule anywhere in the body.

On the contrary to the classical interpretation of limit theorems, a mathematical structure of limit theorems is investigated by the use of Lagrangian duality theory. Suppose that a rigid-plastic boundary value problem shown in figure 1. Our concern is to estimate the ultimate load factor α^* to the reference external load Γ_0. A following Lagrangian L is introduced (Kobayashi, 2003).

$$L\left(\alpha,\boldsymbol{Q},\boldsymbol{Q}^r,s_i,\lambda_i,\boldsymbol{\kappa},\boldsymbol{\mu},t_i\right) = \begin{cases} \alpha+\boldsymbol{\kappa}\cdot\ \ \boldsymbol{B}'^T\boldsymbol{Q}^r\ \ -\sum_i\lambda_i\left(f_i(\boldsymbol{Q})+s_i\right) \\ \quad +\boldsymbol{\mu}\cdot\ \ \boldsymbol{Q}-\alpha\boldsymbol{Q}^E-\boldsymbol{Q}^r\ \ +\sum_i s_i t_i\ \ (\text{for } t_i\geq 0\ ) \\ +\infty \qquad\qquad\qquad\qquad (\text{otherwise}) \end{cases} \tag{1}$$

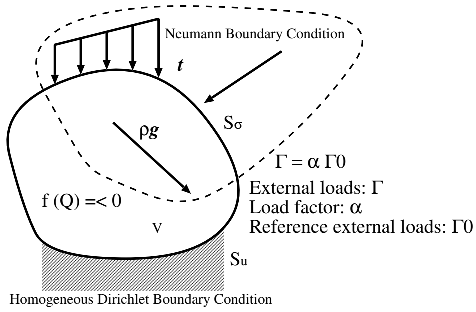

Figure 1. Targeted rigid-plastic boundary value problem

where α is a load factor, the second term is the equilibrium of a residual stress field $\boldsymbol{Q}^r$ which is equilibrating with zero external loads, the third term is no violation of the yielding condition f at each integral points i, the fourth term is a decomposition of a SA stress field $\boldsymbol{Q}$ into the fictitious linear elastic stress field $\boldsymbol{Q}^E$ equilibrating to the reference load Γ_0 and a residual stress field $\boldsymbol{Q}^r$ and the fifth term means the non-negativeness of the slack variable s_i. Variables $\boldsymbol{\kappa}$, λ_i, $\boldsymbol{\mu}$ and t_i are Lagrangian multipliers corresponding to the above mentioned constraint conditions. Lagrangian multipliers $\boldsymbol{\kappa}$, λ_i and $\boldsymbol{\mu}$ can also be interpreted as nodal velocities, plastic multipliers and plastic strain rates, respectively, as is shown later.

The lower bound analysis can be derived the supremum-infimum operation of the Lagrangian L as follows;

$$\sup\left\{\inf\left\{L\,|\lambda_i,\boldsymbol{\kappa},\boldsymbol{\mu},t_i\right\}|\,\boldsymbol{Q}^r,s_i\right\}=\sup\left\{\alpha\,|\boldsymbol{Q},\boldsymbol{Q}^r,s_i\right\}$$
$$\text{Sub.}\quad\begin{cases} f_i(\boldsymbol{Q})+s_i=0,\ s_i\geq 0\ \forall i\in V \\ \boldsymbol{B}'^T\boldsymbol{Q}^r=\boldsymbol{0} \\ \boldsymbol{Q}=\alpha\boldsymbol{Q}^E+\boldsymbol{Q}^r \end{cases} \tag{2}$$

This formulation (2) implies the maximisation of a load factor α subject to the SA and PA stress fields.

Contrary to the lower bound analysis, the upper bound analysis can be derived by the infimum-supremum operation of the Lagrangian L, as shown below.

$$\inf\left\{\sup\left\{L\,|\alpha,\boldsymbol{Q}^r,\boldsymbol{Q},s_i\right\}|\,\boldsymbol{\mu},\boldsymbol{\kappa},\lambda_i\right\}$$
$$=\inf\ \ \bar{D}(\boldsymbol{\mu})\,|\boldsymbol{\mu}\qquad\text{Sub.}\ \begin{cases} 1-\boldsymbol{\mu}\cdot\boldsymbol{Q}^E=0 \\ \boldsymbol{B}'\boldsymbol{\kappa}-\boldsymbol{\mu}=\boldsymbol{0} \\ t_i-\lambda_i=0,\quad t_i\geq 0. \end{cases}, \tag{3}$$

where $\bar{D}(\boldsymbol{\mu})$ is the internal dissipation rate based on following Hill's maximum plastic work principle

$$\inf\left\{D(\boldsymbol{\mu},\lambda_i)\,|\forall\lambda_i\geq 0\right\}$$
$$=\begin{cases}\bar{D}(\boldsymbol{\mu})=\sup\left\{\boldsymbol{\mu}\cdot\boldsymbol{Q}\,|\forall\boldsymbol{Q}\text{ such that } f_i(\boldsymbol{Q})\leq 0\right\}. \\ -\infty \qquad \text{otherwise}\end{cases} \tag{4}$$

In addition to this, the extremal operation of the internal dissipation rate $\bar{D}$ on the stresses $\boldsymbol{Q}$ is equivalent to the associated flow rule;

$$\boldsymbol{\mu}-\sum_i\lambda_i\frac{\partial f_i}{\partial\boldsymbol{Q}}=\boldsymbol{0},\quad\lambda_i\geq 0. \tag{5}$$

In summary, equation (3) means the minimisation of the internal dissipation rate $\bar{D}$ under the conditions of normalisation of the external plastic work rate $\dot{W}_{ext}=\boldsymbol{\mu}\cdot\boldsymbol{Q}^E=1$ and KA and PA velocity fields.

It should be noted that both the upper and lower bound analyses can be derived from the same Lagrangian L just by the change in order of the infimum and supremum operations. Therefore, the upper and lower bound analyses are dual to each other.

Table 1. Mechanical interpretation of a complementarity condition

Slack variable $s_i\,(\geq 0)$	Plastic multiplier $\lambda_i\,(\geq 0)$	Status
0	Positive	Plastic
Positive	0	Rigid
0	0	Neutral

Table 2. Classification of various rigid-plastic methods

	MC	LE	LA	
			UB	LB
Equilibrium	○*1	△*2	×	○
Stress boundary cond.	○	○	×	○
Yielding cond.	○*1	△*3	×	○
Compatibility	×	×	○	×
Velocity boundary cond.	×	×	○	×
Associated flow rule	×	×	○	×
Solution	L*1	?	U	L
Convergence	≈T*4	?	T	T
3 dim. problems	×	○	○	○
Complicated geometry or strength distribution	×	△	○	○

L: Lower bound value, T: True value, U: upper bound value

*1: Incompleteness due to no check in the rigid zone.
*2: Insufficiency, partly satisfied
*3: Insufficiency, no check in each blocks
*4: Incompleteness due to no check in the rigid zone

Furthermore, a duality gap which is a difference between the values of the objective functions of both the upper and lower bound analyses is investigated. After some arrangements, a duality gap is expressed as

$$D\ \ \tilde{\boldsymbol{\mu}}, \tilde{\lambda}_i\ \ -\alpha^0 = \sum_i \tilde{\lambda}_i s_i^0 \geq 0, \qquad \tilde{\lambda}_i \geq 0,\, s_i^0 \geq 0. \tag{6}$$

where a superscript 0 means quantities of the lower bound analysis and ˜ means quantities of the upper bound analysis. A duality gap is always greater than or equal to the scalar products of two non-negative vectors, say slack variables $s_i \geq 0$ and plastic multipliers $\lambda_i \geq 0$. Moreover, according to the duality theorem, a duality gap is zero if and only if the solution is correct. Therefore, following complementarity conditions hold at all the integral points if and only if the solution is correct; $\lambda_i \times s_i = 0$. Mechanical interpretation of the complementarity conditions is summarised in table 1.

3 THEORETICAL CLASSIFICATION OF VARIOUS RIGID-PLASTIC METHODS

Rigid-plastic methods used in the geotechnical engineering can be classified into 3 groups shown in table 2.

One method is a method of characteristics (MC) originally formulated by Kötter. Statical formulation which combines both the equilibrium of forces and the yielding conditions is usually used. Kinematic formulation called Geiringer's equation is also known. Both formulations are dual to each other via the associated flow rule. Therefore, an obtained stress field is SA and PA and its corresponding velocity field is also KA and PA. However, it should be noted that MC discusses the quantities only in the plastic zones and no attention is paid for the quantities in the rigid zones. In this sense, the solutions by MC are sometimes called incomplete solutions.

Another method is limit equilibrium methods (LE). Truly, there are several versions of LE. In the formulation of LE, two ambiguities can be point out as follows. One is the insufficiency of the equilibrium equations. The other is insufficiency of the yielding conditions. These insufficiencies are mainly because only the limited numbers of failure modes are considered. In addition to this, considered failure modes are generally independent of the associated flow rule. As a natural consequence, solutions by LE are generally partly satisfying SA or PA.

The other method is limit analysis (LA) as is explained in details in the previous section. According to the duality theorem, solutions by both the upper and lower bound analyses coincide if and only if the solution is correct.

Applicability of these methods to general rigid-plastic boundary value problems is important for the practical design. Rigid-plastic finite element method (RPFEM) based on limit analysis (LA) can deal with general rigid-plastic boundary value problems systematically. On the other hand, by the manual calculations of LA, limit equilibrium method (LE) or method of characteristics (MC), it is rather difficult to calculated general rigid-plastic boundary value problems including soil-reinforcement interactions, geometric complexity or complicated material distributions. Accordingly, from the point of applicability to general problems, rigid-plastic finite element method (RPFEM) has the advantages.

4 APPLICATION AND EXTENSION OF RIGID-PLASTIC ANALYSIS

4.1 *Hybrid type rigid-plastic finite element method based on the interior point method*

As discussed previously, numerical analysis is inevitable for the application to general rigid-plastic boundary value problems. To this end, rigid-plastic finite element method (RPFEM) (Lee and Kobayashi, 1973, Tamura et al., 1984) which was firstly formulated based on the upper bound theorem. This mighty numerical tool is widely used now especially in the field of metal forming.

On the contrary to this upper bound formulation, the author proposed a hybrid type formulation of rigid-plastic finite element method called *Primal-dual Rigid-plastic finite element method* (PDRPFEM) based on the nonlinear optimisation theory. According to the algorithm of PDRPFEM which belongs to the interior point method (for example, Kojima et al. 2001), a duality gap is gradually reduced to be zero monotonically during the iterative procedure by solving all the constraint conditions simultaneously. That is to say, incremental forms of static constraint conditions (7a), kinematic constraint conditions (7b) and approximated complementarity conditions (7c) are solved simultaneously to modify the assumed solution iteratively;

$$\begin{aligned} &\boldsymbol{Q}^{(k)} = \alpha^{(k)}\boldsymbol{Q}^E + \boldsymbol{Q}^{r\,(k)}, \quad \boldsymbol{B}'^T\boldsymbol{Q}^{r\,(k)} = \boldsymbol{0}, \\ &f_i(\boldsymbol{Q}^{(k)}) + s_i = 0,\, s_i \geq 0, \quad \forall i \in V, \end{aligned} \tag{7a}$$

$$\begin{aligned} &1 - \boldsymbol{\mu}^{(k)} \cdot \boldsymbol{Q}^E = 0 \quad \boldsymbol{B}'\boldsymbol{\kappa}^{(k)} - \boldsymbol{\mu}^{(k)} = \boldsymbol{0}, \\ &\boldsymbol{\mu}^{(k)} - \sum_i \lambda_i \frac{\partial f_i}{\partial \boldsymbol{Q}} = \boldsymbol{0}, \quad \lambda_i \geq 0, \quad \forall i \in V, \end{aligned} \tag{7b}$$

$$\boldsymbol{\Lambda}^{(k)}\boldsymbol{s}^{(k)} = \varepsilon^{(k)}\boldsymbol{e},\, \varepsilon^{(k)} = \quad \frac{\sum_i^p \lambda_i^{(k)} s_i^{(k)}}{p}\Bigg)^{\omega} \tag{7c}$$

where quantities $\boldsymbol{\Lambda}$ and $\boldsymbol{e}$ are defined as $\boldsymbol{\Lambda}^{(k)} = \mathrm{diag}(\lambda_i^{(k)})$ and $\boldsymbol{e} = (1, \cdots 1)$. A scalar ε is called a barrier parameter which a monotonic decreasing series to converge the complementarity conditions, and a scalar ω is a parameter to control the speed of the convergence. A scalar p stands for the total number of integral points in the body. It is emphasised that this formulation may have the numerical advantages for large-scaled problems, because the interior point method is used for the nonlinear optimisation calculations.

For example, numerical results of a bearing capacity problem of a shallow foundation subjected to uniform inclined surface loading shown in figure 2 solved by PDRPFEM are presented (Kobayashi, 2005). Note that a mesh size for RPFEM is enough if it can cover the whole plastic zones.

Numerical results are plotted in figure 3 together with the analytical upper bound solution obtained by Salençon and Pecker

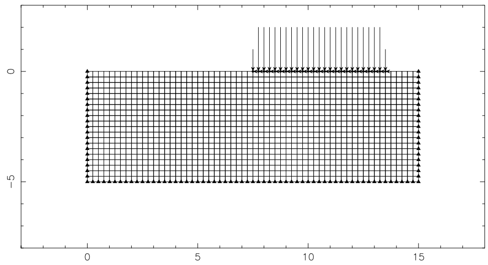

Figure 2. Shallow foundation subjected to uniform inclined loads

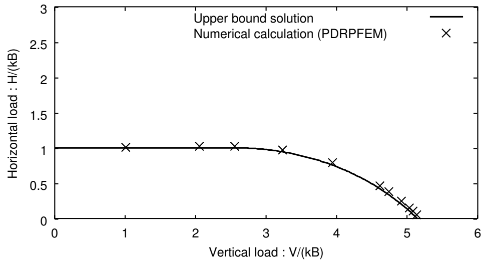

Figure 3. Bearing capacity characteristics (Tresca, Weightless)

(1995). It should be noted that Salençon and Pecker also derived the analytical lower bound solution, which is almost identical to the upper bound solution within the differences of only 0.6%. Numerical results show very good conformation with the analytical solution.

Additionally, calculated failure modes are also discussed. According to the analytical solution, two failure modes are observed depending on the load inclination angle δ. The threshold value of the inclination angle is $\delta^* = 21.3°$. A mass failure mode occurs if the load inclination angle is $\delta < \delta^*$. Otherwise, a slip failure mode is observed. A calculated failure mode in case of $\delta = 5.71°$ is shown in figure 4. Another calculated mode in case of $\delta = 21.8°$ is shown in figure 5. These calculated failure modes also shows quite good conformation to the analytical solution.

According to these results, it may be concluded that PDRPFEM is a reliable numerical tool and it is applicable to numerical plastic design procedure.

4.2 *Extension to shakedown analysis*

Let us consider stability problems subjected to various natural external loads, that is, wind loads, tidal loads, earthquake loads etc. It is advisable to handle these problems in a simple way for practical design. The methodology of limit analysis can be extended to a stability problem of an elastic-plastic body subjected to repeated loads. The theoretical basis of this problem is shakedown theorems (For example, Martin 1975), which can be interpreted as the generalisation of limit theorems. In general, it is difficult to know a detailed time history of the external loads in advance. According to shakedown theorems, we can avoid this difficulty by using shakedown analysis, because only a range of the external loads in a generalised load space is required. This is a great advantage of shakedown analysis for the practical design.

Unfortunately, only few previous researches have been done on the application of shakedown analysis to geotechnical design procedure. Among these few researches, the author has proposed a hybrid type formulation of shakedown analysis. Its algorithm is very similar to the algorithm of PDRPFEM (Kobayashi and Nishikawa, 2004). A simplified analysis by the combination of macro element concept and shakedown analysis is also proposed for a bearing capacity problem of a multi-footing system (Kobayashi and Genjo, 2001).

5 CONCLUSIONS

In this paper, fundamental aspects of rigid plastic analysis are briefly reviewed from the point of mechanics and mathematics. Recent developments of rigid-plastic finite element method

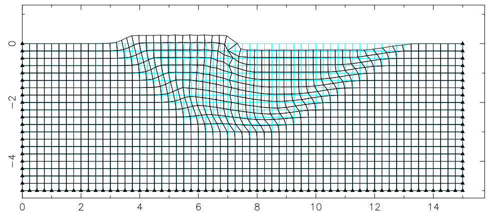

Figure 4. Mass failure mode ($\delta = 5.71°$)

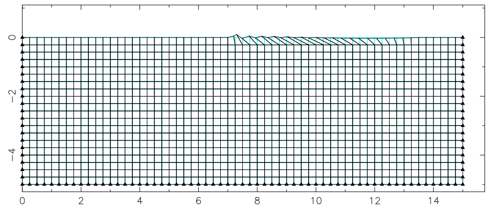

Figure 5. Slip failure mode ($\delta = 21.8°$)

(RPFEM) are also reported. According to these results, the author thinks the time has come to shift the plastic design procedure from the conventional rigid-plastic analysis to the sophisticated numerical analysis. Among the several numerical analyses, rigid-plastic finite element analysis is promising for the practical design, because it is simple but based on the firm theoretical background. It should be emphasised that rigid-plastic analysis is located on the key place between conventional methods and mighty numerical methods shown in figure 6. In this sense, mechanics of rigid-plastic analysis is still important for the modernisation of geotechnical design.

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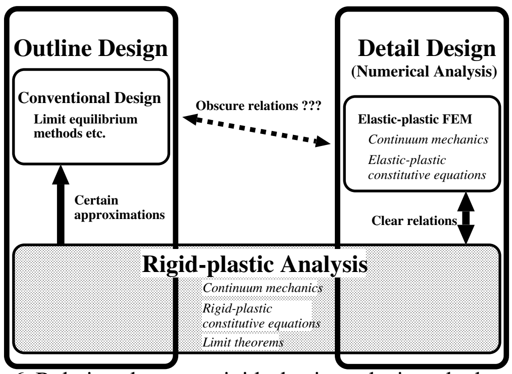

Figure 6. Relations between rigid-plastic analysis and other methods