Proceedings of the 16th International Conference on Soil Mechanics and Geotechnical Engineering

doi:10.3233/978-1-61499-656-9-2845

Geotechnical reliability analyses: towards development of some user-friendly tools

Analyse de fiabilité géotechnique: vers le développement d'outils conviviaux

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ABSTRACT
This paper discusses two powerful and yet user-friendly reliability techniques that could be implemented on the PC easily by practitioners. The techniques are representation of non-Gaussian random variables using Hermite polynomials and subset Markov chain Monte Carlo simulation. It is believed that ease of implementation would popularize use of reliability-based design in practice.

RÉSUMÉ
Cet article expose deux méthodes puissantes mais faciles d'emploi d'analyse de fiabilité. Ces méthodes peuvent être facilement implantées sur un ordinateur personnel par des praticiens non expérimentés. Les deux méthodes sont la représentation de variables aléatoires non gaussiennes à l'aide de polynôme d'Hermite et la simulation Monte Carlo de sous-ensemble de chaîne de Markov. Les auteurs sont d'avis que leur facilité d'implantation puisse favoriser la démocratisation de l'emploi des méthodes de fiabilité dans la pratique de la géotechnique.

1 INTRODUCTION

The advent of the World Trade Organisation (WTO) has added impetus to the formation of trading groups that result in multilateral free trade areas or bilateral free trade agreements. Traditionally, geotechnical engineering practice has always been viewed as a localised activity under the purview of the relevant federal and/or state authorities. However, the move towards greater economic cooperation and integration will require the elimination of some technical obstacles that exist as a consequence of differences in national codes and standards, and harmonization of technical specifications.

One on-going example of this code harmonisation phenomenon is the development of the Eurocodes within the European Union. The Eurocodes were initiated by the Commission of the European Communities (CEC) as a development of the Construction Products Directive that requires a series of harmonised European Standards to provide certain "Essential Requirements" of safety, economy, and fitness for use but which do not hinder trade within the European Community (Simpson and Driscoll, 1998). It is quite clear that the focus of harmonisation is to facilitate trade and perhaps more importantly, to ensure fair competition in the construction industry between member nations. Similar harmonisation efforts currently are underway in Canada (Green and Becker, 2001) and Japan (Honjo and Kusakabe, 2002).

The system of Structural Eurocodes consists of 10 standards. The basis for design is laid out in the head Eurocode EN1990:2002 that describes the principles and requirements for safety, serviceability and durability of structures, the basis for their design and verification, and gives guidelines for related aspects of structural reliability. It suffices to note here that reliability analysis is the only available theoretical tool capable of ensuring consistent safety between different materials. Although Eurocode 7 (Geotechnical design) adheres to the limit state design concept, it is not as well integrated as the other material codes because its partial factors are essentially empirical and precedence-based, rather than reliability-based. The important point here is that code harmonization also entails harmonization between structural and geotechnical design. Because geotechnical design is only one component of the Structural Eurocodes, it is anticipated that structural reliability methods will eventually prevail in Eurocode 7.

There is an urgent need for geotechnical engineers to play a more active role in the development of reliability-based design (RBD) methodology. Currently, most of the impetus in the development of RBD codes arises from within the structural engineering community. However, research in geotechnical reliability that addresses issues relevant to the geo-profession is progressing quite rapidly in recent years. Three specialty workshops have been organized over the past 3 years. They are the International Workshop on Foundation Design Codes and Soil Investigation in view of International Harmonization and Performance Based Design (Honjo et al., 2002), International Workshop on Limit State design in Geotechnical Engineering Practice (Phoon et al., 2003) and International Workshop on Risk Assessment in Site Characterization and Geotechnical Design (Sivakumar Babu and Phoon, 2004). A more complete list of past activities from 1971 is given at www.geoengineer.org/reliability.

Despite emerging indications that regulatory pressure will eventually bring geotechnical design within a reliability framework and the growing interest in the research community, it is accurate to say that the average practitioner is largely unfamiliar with RBD and its potential benefits. One reason is that most research papers do not provide computational steps with sufficient details for the lay person to implement reliability analysis using common desktop softwares. In short, there is a lack of user-friendly tools. Low and Tang (2004) demonstrated that simple geotechnical reliability problems could be analyzed easily using EXCEL, but their emphasis on pedagogy is rare within a research culture that values originality. The objective of this paper is to provide a glimpse of two powerful reliability techniques that could be readily implemented on a modest PC. Section 2 discusses how non-Gaussian random variables can be represented using Hermite polynomials in a completely general way, even if the data do not fit any known classical probability distributions. Extension to correlated random vectors is quite straightforward but covered elsewhere (Phoon and Nadim, 2004). Section 3 discusses how small probabilities of failure can be computed efficiently using the Markov chain Monte Carlo simulation method, which is more powerful and general than the popular First-Order Reliability Method (FORM).

2 FORM USING HERMITE POLYNOMIALS

2.1 *Hermite polynomials*

Hermite polynomials are given by:

$$\begin{aligned} H_0(U) &= 1 \\ H_1(U) &= U \\ H_2(U) &= U^2 - 1 \\ H_3(U) &= U^3 - 3U \\ H_{k+1}(U) &= U\, H_k(U) - k\, H_{k-1}(U) \end{aligned} \tag{1}$$

where U is a standard Gaussian random variable. The last row of Eq. (1) shows that Hermite polynomials of any degree (k) can be computed efficiently using a simple recurrence relation that depends only on two preceding Hermite polynomials. This recurrence relation can be implemented directly using EXCEL. It can be proven rigorously (Phoon, 2003) that any random variable X (with finite variance) can be expanded as follows:

$$X = \sum_{k=0}^{\infty} a_k H_k(U) \tag{2}$$

The numerical values of the coefficients, a_k, depend on the distribution of X. They can also be computed readily using EXCEL even if the empirical distribution of X cannot be fitted to any classical probability distribution functions as follows (Berveiller et al., 2004):

1. Let $\underline{x}$ be a $n \times 1$ vector containing measured data or simulated data from a known cumulative distribution function, Q(x).
2. Let $\underline{u} = \Phi^{-1}Q(\underline{x})$ be a $n \times 1$ vector containing n realizations of a standard Gaussian random variable. The function Φ^{-1} can be invoked using NORMSINV in EXCEL.
3. Let $\underline{h}_0$ be a $n \times 1$ vector containing ones, $\underline{h}_1 = \underline{u}$, $\underline{h}_2 = \underline{u}\ .\!*\underline{u} - 1$, … $\underline{h}_{p-1} = \underline{u}.*\underline{h}_{p-2} - (p\text{-}2)\underline{h}_{p-3}$, and H be a $n \times p$ matrix containing $\underline{h}_0$, $\underline{h}_1$, $\underline{h}_2$, … $\underline{h}_{p-1}$ in the columns. The operator ".*" means element-wise matrix multiplication (MATLAB convention), i.e., for matrix A = B.*C, (i, j) element in A, $a_{ij} = b_{ij} \times c_{ij}$.
4. Let $\underline{a}$ be a $p \times 1$ vector containing the unknown Hermite coefficients $\{a_0, a_1, a_2, \ldots a_{p-1}\}^T$. This vector is computed by solving the following system of linear equations:

   $$(H^T H)\underline{a} = H^T \underline{x} \tag{3}$$

   Eq. (3) can be solved easily using array formulae and matrix functions in EXCEL (TRANSPOSE, MMULT, MINVERSE).

2.2 *Convergence in probability tails*

The gamma probability density function is given by:

$$q(x) = \frac{1}{b^c \Gamma(c)} x^{c-1} e^{-\frac{x}{b}} \tag{4}$$

where b is the scale parameter and c is the shape parameter. Fig. 1 shows two gamma distributions with $b = 1$, $c = 0.5$ (top) and $b = 1$, $c = 2$ (bottom). The former distribution is related to a chi-square distribution with one degree of freedom. Although it is highly skewed, a 6-term Hermite expansion is sufficient to match probabilities as low as 10^{-4}. It is possible to match even smaller probabilities by increasing the simulation sample size beyond 50000, but 10^{-4} is sufficient for most reliability problems. The second gamma distribution looks "Gausian" and it is not surprising that an even shorter 4-term Hermite expansion is enough for this case.

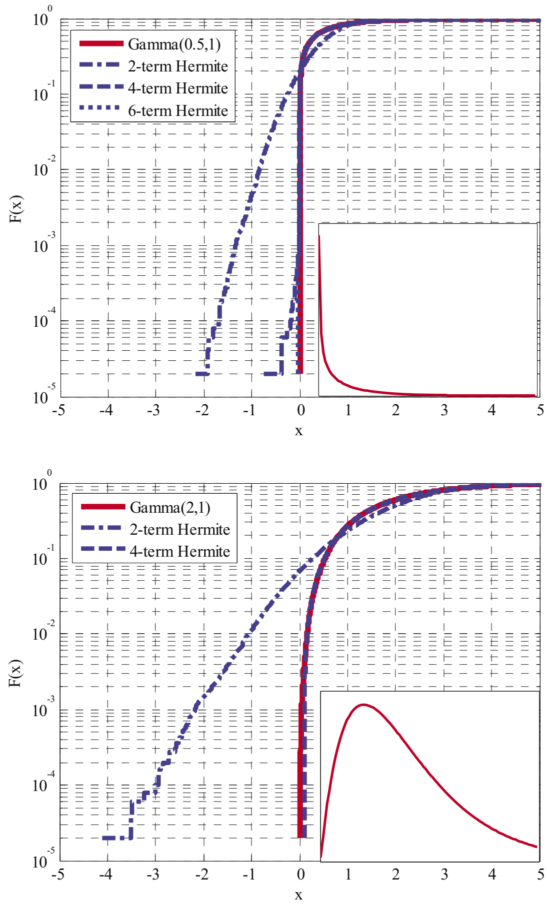

Figure. 1 Cumulative distribution function of Hermite expansion for two gamma distributions (simulation sample size = 50000).

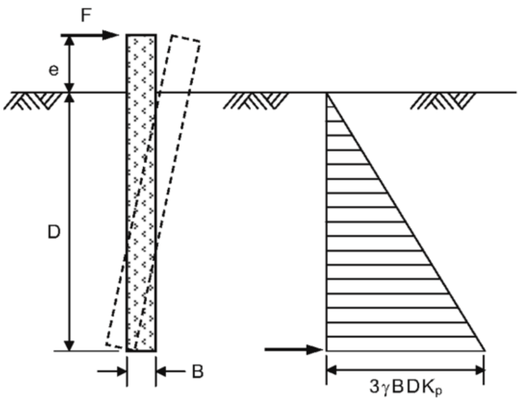

Figure 2. Simplified Broms approach for laterally-loaded free-head rigid pile in sand.

2.3 *Application in FORM*

This section demonstrates that FORM used in conjunction with Hermite polynomials is computationally more robust than the commonly used equivalent Gaussian technique. The reason is that FORM iterations can take place fully in standard Gaussian space, where all variables are scaled to unit variance. The example considered is a laterally-loaded free-head rigid pile in sand (Fig. 2). The performance function (G) is given by:

$$G = 0.5M \frac{\gamma B D^3 \tan^2(45^\circ + \phi'/2)}{(e+D)} - F \tag{5}$$

where M is the random factor describing the model uncertainty, B and D are respectively the diameter and length of the pile, e

is the eccentricity, γ is the unit weight of sand, ϕ' is the effective friction angle, and F is the applied load. In this example, M is modeled as a Gamma random variable with mean = 1.3 and standard deviation = 0.5. Hence, the scale and shape parameters are b_M = 0.192 and c_M = 6.760. The applied load F is also assumed to be Gamma distributed with mean = 1000 kN and standard deviation = 250 kN (or b_F = 62.5 and c_F = 16.0). The rest of the parameters were assumed to be deterministic and were given by: D = 10 m, B = 1 m, e = 1 m, γ = 18 kN/m^3 and ϕ' = 40°. The mean factor of safety is 4.9.

The First-Order Reliability Method (FORM) computes β as follows:

$$\beta = \min \sqrt{u_1^2 + u_2^2} \qquad \text{subject to } G \le 0 \tag{6}$$

Using the equivalent Gaussian technique, the standard Gaussian variates (u_1 and u_2) can be evaluated as follows:

$$u_1 = \frac{m - \mu_M^N}{\sigma_M^N} = \frac{m - \{m - \sigma_M^N \Phi^{-1}[Q_M(x)]\}}{\phi\{\Phi^{-1}[Q_M(x)]\} / q_M(x)} \tag{7}$$

$$u_2 = \frac{f - \mu_F^N}{\sigma_F^N} = \frac{f - \{f - \sigma_F^N \Phi^{-1}[Q_F(x)]\}}{\phi\{\Phi^{-1}[Q_F(x)]\} / q_F(x)} \tag{8}$$

where m and f are the respective trial values of the Gamma variables M and F. By substituting Eqs. (7) and (8) into Eq. (6), it can be seen that β is computed by nonlinear optimization in the original space. In practice, this can be easily done using the Solver function in EXCEL. Direct application of Solver to this example produces an incorrect answer of β = 3.133. The reason is that variables in original space are not properly scaled. The values of f are about three orders of magnitude larger than the values of m. The correct answer β = 2.933 can only be obtained if one applies “Use Automatic Scaling” under Solver Options. However, if one initiates FORM iterations at m = 1.3 and f = 800, Solver would produce an error (even with automatic scaling). The solution in the original space is clearly not robust.

In the Hermite polynomial method, FORM iterations can take place fully in standard Gaussian space. The standard Gaussian variates (u_1 and u_2) at each iteration are converted to m and f using Eqs. (2) and (3). Results for various Hermite expansions are summarized in Table 1 (simulation sample size for Hermite coefficients n = 100). A two-term and four-term Hermite expansion for F and M are sufficient in this example. The longer expansion is needed for M because the coefficient of variation is larger. This example demonstrates that representing non-Gaussian random variables using Hermite expansions in FORM is very robust from a computational viewpoint. It is not necessary to “Use Automatic Scaling” in Solver and the same solution is obtained regardless of the initial starting values for u_1 and u_2.

3 SUBSET MCMC SIMULATION

3.1 *Background*

One of the user-friendly methods to carry out geotechnical reliability analysis is the Monte Carlo simulation (MCS) technique. Due to the rapid increase in desktop computational power, this method is becoming very attractive. Compared to FORM, this method is much easier to implement in computer programs, especially for cases where the performance functions are complex non-linear functions of basic variables. It is also more intuitive to the lay person.

Numerous methods have been developed to improve the calculation efficiency of the MCS technique. These include importance sampling, stratified sampling, Latin hypercube sampling etc. (e.g. Schüeller et al., 1989). These methods, however, are generally rather problem dependent and accurate solutions are difficult to obtain without knowing the detailed features of each technique employed.

There are new findings and applications in MCS technique recently from the areas of computational statistics and financial engineering including Markov chain Monte Carlo (MCMC) technique (Gilks et al., 1998) and low discrepancy sequences. The application of MCMC to reliability analysis was proposed by Au and Beck (2003), which is briefly summarized below.

3.2 *Subset method and MCMC*

Subsets: Let the failure region be F and the failure probability be P_F. The whole region is denoted by F_0 and its subsets are denoted by F_i where $F_m = F$ is assumed. Thus, the following relationship holds (Fig. 3):

$$F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_m = F \tag{9}$$

The failure probability can be calculated based on these subsets as follows:

$$P_F = P(F_m) = P(F_m | F_{m-1}) P(F_{m-1} | F_{m-2}) \cdots P(F_1 | F_0) \tag{10}$$

MCMC (Markov Chain Monte Carlo) method: In order to use the subsets defined above effectively, one need to generate samples for any PDF defined on subset F_i. This can be done by using MCMC. One MCMC algorithm known as the Metropolis-Hastings (M-H) algorithm is described below.

M-H algorithm generate Markov chain samples $x^{(t)}$ to $x^{(t+1)}$ that follows a target PDF $\pi(x)$. $x^{(t+1)}$ is generated based on $x^{(t)}$ by the procedure below:

(1) Select a proposal density function $q(x' | x^{(t)})$.

(2) Calculate an acceptance probability α_t as:

$$\alpha_t = \min\left\{1, \frac{q(x^{(t)} \mid x')\pi(x')}{q(x' \mid x^{(t)})\pi(x^{(t)})}\right\} \tag{11}$$

(3) $x^{(t+1)}$ is generated based on α_t as:

$$x^{(t+1)} = \begin{cases} x' & \alpha_t \\ x^{(t)} & 1 - \alpha_t \end{cases} \tag{12}$$

We set $x^{(t+1)} = x'$ with probability α_t. Otherwise, it is set to $x^{(t)}$. It is not difficult to prove that $x^{(t)}$ follows PDF $\pi(x)$ under some mild regularity conditions (Gilks et al., 1998). The essence here, however, is that by using MCMC, one can generate samples based on any PDF like the conditional PDF defined in a subset.

3.3 *Failure probability calculation by subset MCMC*

Based on the subsets defined in Eq. (9) and MCMC, the failure probability can be calculated using the following procedure:

(1) Generate N_t samples from a given PDF.

Table 1: FORM solutions using Hermite expansions of various lengths for Gamma distributed F and M.

No. of terms F	No. of terms M	Reliability index	Probability of failure
	2	2.084	1.86×10^{-2}
2	4	2.942	1.63×10^{-3}
	6	2.933	1.67×10^{-3}
	2	2.094	1.81×10^{-2}
4	4	2.940	1.64×10^{-3}
	6	2.933	1.68×10^{-3}
Equivalent Gaussian		2.933	1.68×10^{-3}

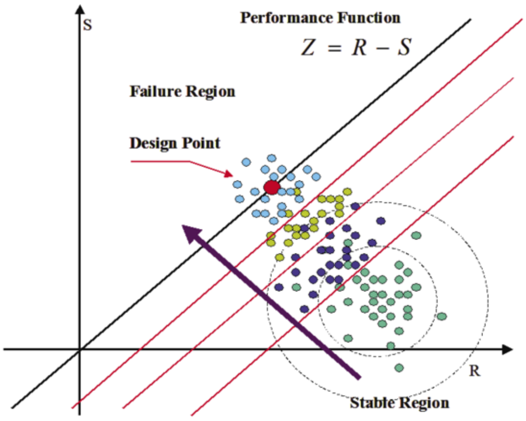

Figure 3. Concept of the subset MCMC method.

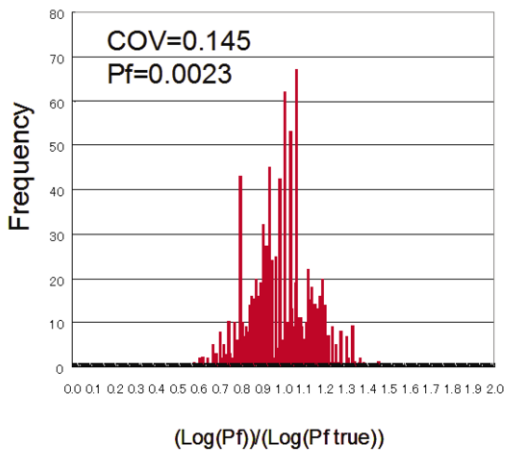

Figure 4. Results of a subset MCMC example.

(2) Define subset F_{k+1} by using N_s ($< N_t$) samples that are closer to the limit state $Z=0$, where F_{k+1} is defined as:

$$F_{k+1} = \left\{ x \,\middle|\, Z(x) \leq \frac{Z_{N_s} + Z_{N_s+1}}{2} \right\} \tag{13}$$

In subset F_k, the probability that samples in F_{k+1} are generated is N_s/N_t. Note that Z_i are ordered generated samples from smaller to larger Z.

(3) By using MCMC, it is possible to generate N_t samples within F_{k+1}.

(4) When sufficient number of failure cases is generated at step m, the simulation is stopped; otherwise go back to step (2) to continue the calculation. If the simulation is stopped, P_F is given by the following equation:

$$P_F = P(Z \leq 0) = \left(\frac{N_s}{N_t} \right)^{m-1} \frac{N_f}{N_t} \tag{14}$$

where N_f is number of samples $Z<0$ (failure).

3.4 *An example calclulation*

An example calculation is presented to illustrate the methodology. A simple performance function consisting of two unit variance normal random variables R and S with respective means 7.0 and 3.0 is defined as follows:

$$Z = R - S \tag{15}$$

The exact failure probability, P_F, for this case is equal to 0.002339. The total number of generated sample for each subset, N_t, is set to 100, and the number of samples chosen to be used in the next step, N_s, is set to 10 in this case. Some experience is necessary to choose these N_t and N_s for each problem.

The result of this simulation is presented in Fig. 4 where 1000 trials have been carried out. The coefficient of variation of the estimated $log(P_F)$ for this case is 0.145, which implies that estimated P_F is a sufficiently accurate for practical reliability analyses.

The practical advantages of the subset MCMC are its efficiency as well as its relatively automated calculation procedure where very little judgment is involved.

4 CONCLUSION

The necessity of introducing RBD in geotechnical engineering is emphasized but there is a lack of user-friendly and powerful computational tools to popularize reliability analysis in practice. The average practitioner does not have the time or the inclination to compute reliability indices by learning complex theories from scratch. Two recent developments in this area are briefly introduced, namely representation of non-Gaussian random variables using Hermite polynomials and subset MCMC technique. The authors believe that it is crucial to explain the potential of these developments in a simple and algorithmic manner so that reliability analyses of complex real-world problems are within reach of the average practitioners. This is an important step towards establishing geotechnical design methods that could achieve more appropriate and consistent safety levels.

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