*ECAI 2014*
*T. Schaub et al. (Eds.)*


*doi:10.3233/978-1-61499-419-0-87*

# Incremental Elicitation of Choquet Capacities for Multicriteria Decision Making

**Nawal Benabbou** and **Patrice Perny** and **Paolo Viappiani** [1,2]

**Abstract.** The Choquet integral is one of the most sophisticated and expressive preference models used in decision theory for multicriteria decision making. It performs a weighted aggregation of criterion values using a capacity function assigning a weight to any coalition of criteria, thus enabling positive and/or negative interactions among criteria and covering an important range of possible decision behaviors. However, the specification of the capacity involves many parameters which raises challenging questions, both in terms of elicitation burden and guarantee on the quality of the final recommendation. In this paper, we investigate the incremental elicitation of the capacity through a sequence of preference queries selected one-by-one using a minimax regret strategy so as to progressively reduce the set of possible capacities until a decision can be made. We propose a new approach designed to efficiently compute minimax regret for the Choquet model. Numerical experiments are provided to demonstrate the practical efficiency of our approach.

## 1 INTRODUCTION

In the field of Multicritieria Decision Making, aggregation functions are often used to compare alternatives evaluated on multiple conflicting criteria by synthesizing their performances into overall utility values. Such functions must be sufficiently expressive to fit to Decision Maker's (DM) preferences, allowing for instance the determinination of his/her preferred alternative. Choquet integrals form a family of non-linear aggregators that are really appealing for preference modeling because they enable to model different kind of interactions between criteria and include many aggregators as special cases (e.g. linear additive models, min, max and any other order statistics, leximin and leximax, OWA and WOWA [24, 29] and Yaari's model [28]). Choquet integral has received much attention in the last decades and is now widely used in practical decision making [9].

However, to compute overall utility values using a Choquet integral, decision support systems need to be able to assess model's parameters according to DM's preferences. These parameters used to capture the value system of the DM are characterized by a capacity function defining the weight attached to every subset of criteria. Therefore, they are in exponential number relatively to the number of criteria and their elicitation is a challenging issue. Most of previous works on capacity elicitation for Choquet integrals consider a static preference database as input, and focus on the determination of a set of capacity values that best fits the available preferences. For example, one can minimize a quadratic error between Choquet values and target utility values prescribed by the DM on a sample of reference alternatives. Alternatively, one can impose some constraints on Choquet values to enforce the decision model to be compatible with a partial or total order available on a subset of alternatives. These approaches are illustrated in many papers see, e.g. [11, 18, 19, 22, 13] and [10] (Chapter 11), some of them being implemented in decision support softwares such as TOMASO [15] and MYRIAD [12].

Departing from these standard approaches, we are proposing in this paper an incremental elicitation process for Choquet integrals, in which preference queries are selected one at a time, to be as informative as possible, so as to progressively reduce the set of admissible capacities until a robust recommendation can be made. This approach relies on and extends previous works on the incremental elicitation of linear utility functions, going back to the ISMAUT method [14] and more recently, strategies developed within the artificial intelligence community for preference query selection using the minimax-regret criterion [6, 2, 27, 3]. Regret-based elicitation has been successfully demonstrated with real users in a prototype for decision support (UTpref) and validated in a user study [5]. Adaptation of minimax regret elicitation strategies to Choquet models is not obvious, as the number of constraints required to impose that the parameters of the model are valid is exponential in the number of criteria, in the general case. In this paper we propose an efficient algorithm that avoids this issue by focusing on specific (but intuitive) types of preference statements.

## 2 BACKGROUND AND NOTATION

Let $\mathcal{X}$ be the set of alternatives (items, products, candidates...) that need to be compared in order to make a decision. Any alternative $x \in \mathcal{X}$ is evaluated with respect to a set of $n$ criteria denoted $N = \{1, \ldots, n\}$, and is characterized by a performance vector $(x_1, \ldots, x_n)$ where $x_i \in [0, 1]$ represents the utility of $x$ with respect to the criterion $i$ for all $i \in N$. For simplicity, $x$ will indifferently denote the alternative or its performance vector.

### 2.1 Choquet integrals

For any alternative $x \in \mathcal{X}$, let $(.)$ denote the permutation of $\{1, \cdots, n\}$ which sorts the components of $x$ by increasing order, i.e. $x_{(i)} \leq x_{(i+1)}$ for $i \in [\![1, n-1]\!]$. Let $X_{(i)}$ denote the subset of criteria with respect to which $x$ has a utility greater or equal to $x_{(i)}$, i.e. $X_{(i)} = \{(i), \ldots, (n)\}$; note that $X_{(i+1)} \subseteq X_{(i)}$ for all $i \in [\![1, n-1]\!]$ by definition. In the sequel, $X_{(i)}$ will be referred to as the $i^{th}$ *level set* of $x$ and $Y_{(i)}$ will denote the $i^{th}$ *level set* of an alternative $y \in \mathcal{X}$. Let $v$ be a Choquet capacity, i.e. a real-valued set function defined on $2^N$ such that $v(\emptyset) = 0, v(N) = 1$ and $v(A) \leq v(B)$ for all $A \subseteq B \subseteq N$, $v(A)$ representing the weight attached to coalition $A$, for any $A \subseteq N$. The Choquet integral is defined by:

[1] Sorbonne Universités, UPMC Univ Paris 06, UMR 7606, LIP6, 4 Place Jussieu, 75005 Paris, France; email: name.surname@lip6.fr
[2] CNRS, UMR 7606, LIP6, F-75005, Paris, France.

$$C_v(x) = \sum_{i=1}^{n} \left[x_{(i)} - x_{(i-1)}\right] v(X_{(i)}) \text{ with } x_{(0)} = 0$$

Hence an alternative $x$ is as least as good as $y$ whenever $C_v(x) \geq C_v(y)$. For example, consider a problem defined on 3 criteria, i.e. $N = \{1,2,3\}$, and two performance vectors $x = (0.7, 0.6, 1)$ and $y = (0.8, 1, 0.6)$. The computation of their Choquet value with the following capacity gives:

| | $\emptyset$ | $\{1\}$ | $\{2\}$ | $\{3\}$ | $\{1,2\}$ | $\{1,3\}$ | $\{2,3\}$ | $\{1,2,3\}$ |
|---|---|---|---|---|---|---|---|---|
| $v$ | 0 | 0.1 | 0.2 | 0.3 | 0.5 | 0.6 | 0.7 | 1 |

$$C_v(x) = 0.6 + (0.7 - 0.6)v(\{1,3\}) + (1 - 0.7)v(\{3\}) = 0.75$$
$$C_v(y) = 0.6 + (0.8 - 0.6)v(\{1,2\}) + (1 - 0.8)v(\{2\}) = 0.74$$

Hence we have $C_v(x) > C_v(y)$, meaning that $x$ is strictly preferred to $y$. In multicriteria decision making, one needs to ensure that $C_v(x) \geq C_v(y)$ whenever $x$ weakly Pareto-dominates $y$ (i.e. $x_i \geq y_i$ for all $i \in N$). This property holds due to the monotonicity of $v$ with respect to set inclusion.

In many papers on multicriteria optimization with a Choquet integral, the capacity is assumed to be given [8, 23, 7, 17]. This assumes that preference elicitation methods are available to determine the capacity that best fits DM's preferences. Following the line opened by Boutilier in [27, 3] for simpler decision criteria, we suggest here adopting an incremental approach where capacity elicitation is seen as a game played with the DM. At every step of the elicitation process, the system generates a preference query, and then the DM reveals a piece of his/her actual preferences. The answer provides new constraints on the set of admissible capacities thus reducing the uncertainty attached to the capacity and therefore to the Choquet values. In this process, both the problem of selecting the next query and the one of generating a recommendation are seen as a decision problem under uncertainty, where the uncertainty is due to the imperfect knowledge of preference parameters (here the capacity). Our strategy to select the most promising alternative is based on a minimax criterion aiming at providing the best Choquet value against all admissible choices for the capacity. The selection of the query is made so that an effective regret reduction is guaranteed whatever the answer is. We present now more formally this approach.

## 2.2 Minimax regret criterion

Minimax regret [21, 16] is a decision criterion classicaly used for optimization under uncertainty over data; it has been more recently advocated for use in decision-making where the uncertainty is over utility values [3, 20]. Let $\mathcal{P}$ be a set of preference statements that match with the preferences of the DM. $\mathcal{P}$ can include different types of information, from prior knowledge to information obtained by asking queries to the DM. Assume that DM's preferences can be modeled by an element in the family of aggregators $\mathcal{F}_\Theta$, where $\Theta$ is the set of all admissible parameters of the family (e.g. whenever $\mathcal{F}_\Theta$ represents the family of Choquet integrals, then $\Theta$ is the set of all possible capacities). Let $\Theta_\mathcal{P}$ denote all parameters in $\Theta$ compatible with $\mathcal{P}$.

**Definition 1.** *Given a set of preferences $\mathcal{P}$ and assuming a model $\mathcal{F}_\Theta$, the pairwise max regret of the alternative $x$ with respect to the alternative $y$ is defined as follows:*

$$\mathrm{PMR}(x, y; \Theta_\mathcal{P}) = \max_{\theta \in \Theta_\mathcal{P}} f_\theta(y) - f_\theta(x)$$

*where $f_\theta \in \mathcal{F}_\Theta$ is the aggregator corresponding to parameter $\theta$.*

In other words, the pairwise max regret of $x$ with respect to $y$ represents the worst-case loss when recommending $x$ instead of $y$.

**Definition 2.** *Given a set of alternatives $\mathcal{X}$, a set of preferences $\mathcal{P}$ and assuming a model $\mathcal{F}_\Theta$, the max regret of $x \in \mathcal{X}$ is defined as:*

$$\mathrm{MR}(x, \mathcal{X}; \Theta_\mathcal{P}) = \max_{y \in \mathcal{X}} \mathrm{PMR}(x, y; \Theta_\mathcal{P})$$

In other words, the max regret of $x$ is the worst-case loss when recommending $x$ instead of one of the adversary's choices (i.e. $\arg\max_{y \in \mathcal{X}} \mathrm{PMR}(x, y; \Theta_\mathcal{P})$).

**Definition 3.** *Given a set of alternatives $\mathcal{X}$, a set of preferences $\mathcal{P}$ and assuming a model $\mathcal{F}_\Theta$, the minimax regret is defined as:*

$$\mathrm{MMR}(\mathcal{X}; \Theta_\mathcal{P}) = \min_{x \in \mathcal{X}} \mathrm{MR}(x, \mathcal{X}; \Theta_\mathcal{P})$$

An optimal solution for the minimax regret criterion is an alternative that achieves the minimax regret (i.e. $\arg\min_{x \in \mathcal{X}} \mathrm{MR}(x; \Theta_\mathcal{P})$). Recommending the latter alternative allows one to guarantee that the worst-case loss is minimized. In the rest of this article, $x^*$ will denote one optimal solution for the latter criterion and $y^*$ one of its adversary's choices, arbitrary chosen in $\arg\max_{y \in \mathcal{X}} \mathrm{PMR}(x^*, y; \Theta_\mathcal{P})$.

## 2.3 Incremental elicitation

Given a particular set of preference statements, the worst-case loss ensured by the minimax regret criterion might still be at unacceptable level. By considering additional preferences statements (inducing constraints on the set of admissible parameters), this loss may be decreased. Indeed, we know that $\Theta_{\mathcal{P}'} \subseteq \Theta_\mathcal{P}$ for any set of preference statements $\mathcal{P}' \supseteq \mathcal{P}$; then, $\mathrm{PMR}(x, y; \Theta_{\mathcal{P}'}) \leq \mathrm{PMR}(x, y; \Theta_\mathcal{P})$ for any $x, y \in \mathcal{X}$, and so $\mathrm{MR}(x, \mathcal{X}; \Theta_{\mathcal{P}'}) \leq \mathrm{MR}(x, \mathcal{X}; \Theta_\mathcal{P})$ for any $x \in \mathcal{X}$. Finally, $\mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P}'}) \leq \mathrm{MMR}(\mathcal{X}; \Theta_\mathcal{P})$ and so the minimax regret cannot increase by adding preference statements (usually it decreases (see [4], pp. 194-202)). Therefore, the minimax regret criterion can be used within an incremental elicitation process that progressively asks preference queries to the DM until the minimax regret drops under a given threshold. At that time, recommending $x^*$ ensures that the loss incurred by not choosing the true optimal alternative is bounded above by that threshold.

Different types of queries can be used when designing such incremental elicitation process. Comparisons queries are relatively simple, they require the DM to compare a pair of alternatives and state which one is preferred. Notice however that some queries are more informative than others (e.g. minimax regret won't decrease when asking to compare an alternative with another that Pareto-dominates the former). Thus, it is important to make a good recommendation without asking too many queries, focusing on relevant queries. A notion of myopic value of information can be used [25] to evaluate the relevance of a query. Let $\mathcal{Q}$ denote the set of all considered queries.

**Definition 4.** *Given a set of alternatives $\mathcal{X}$, a set of preferences $\mathcal{P}$ and assuming a model $\mathcal{F}_\Theta$, the worst-case minimax regret of a query $q \in \mathcal{Q}$ is defined as follows:*

$$\mathrm{WMMR}(q, \mathcal{X}; \Theta_\mathcal{P}) = \max_{p \in \mathcal{P}_q} \mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P} \cup \{p\}})$$

*where $\mathcal{P}_q$ denotes the set of all possible answers to the query $q$.*

Hence the next query of the elicitation process should be chosen in $\arg\min_{q \in \mathcal{Q}} \mathrm{WMMR}(q, \mathcal{X}; \Theta_\mathcal{P})$ because any optimal solution for the WMMR criterion ensures the best reduction of minimax regret in the answer's worst-case scenario. Note that computing the optimal query for WMMR can be computationally intensive when set $\mathcal{Q}$ under consideration is too large. We discuss now computational issues related to minimax regret optimization for Choquet integrals (Section 3). We will present our strategy for generating queries within an incremental elicitation process in Section 4.

# 3 MINIMAX REGRET OPTIMIZATION FOR CHOQUET INTEGRALS

In the procedure defined in subsection 2.2, we have to compute $\mathrm{PMR}$ for all ordered pairs of distinct items to determine the current optimal alternative $x^*$; notice that $\mathrm{PMR}(x,x)=0$ and that, in general, $\mathrm{PMR}(x,y) \neq \mathrm{PMR}(y,x)$. Then, the maximum regret $\mathrm{MR}$ can be computed for each alternative so as to determine the item having the lowest $\mathrm{MR}$. However, the computational effort can be significantly reduced using standard pruning rules for min aggregators, as shown in [4]. Empirically, if we use such pruning rules, the number of $\mathrm{PMR}$ computations is only slightly higher than linear, but of course remains quadratic in the worst-case.

We therefore focus our discussion on the computation of $\mathrm{PMR}$, assuming $\mathcal{F}_\Theta$ is the set of Choquet Integrals. In that case, $\Theta_\mathcal{P}$ is the set of all capacities compatible with $\mathcal{P}$.

## 3.1 A General Optimization of Pairwise Max Regret using Linear Programming

Let $v: 2^N \to \mathcal{R}$ be a set function and $v_A$ the decision variable representing $v(A)$ for any $A \subseteq N$. Using this notation, $v$ will indifferently denote the set-function and the vector composed of its values. Thus, for any alternatives $x, y \in \mathcal{X}$, $\mathrm{PMR}(x,y;\Theta_\mathcal{P})$ can be computed by solving the following linear program:

$$\begin{aligned}
\max_v \quad & C_v(y) - C_v(x) && (1)\\
s.t. \quad & v_\emptyset = 0 && (2)\\
& v_N = 1 && (3)\\
& v_A \leq v_{A\cup\{i\}} \quad \forall A \subset N,\ \forall i \in N \backslash A && (4)\\
& C_v(a) \geq C_v(b) \quad \forall a,b \text{ s.t } a \succsim b \in \mathcal{P} && (5)
\end{aligned}$$

Equations (2-4) ensure that $v$ is indeed a capacity and Equation (5) ensures that $v$ is compatible with $\mathcal{P}$. Thus, for Choquet integrals, the computation of $\mathrm{PMR}$ involves exponentially many variables and monotonicity constraints (4). For some specific subclasses of capacities (e.g. 2-additive capacities [22]), it has been shown that the number of such constraints that are actually needed is much lower. However, these subclasses correspond to specific attitudes that do not necessarily match with the observed preferences. Hence, we investigate now the general case without any prior restriction on the admissible set of capacities.

## 3.2 A linear programming formulation for $1A0 \succsim \Lambda$ preference statements

For any two performance vectors $x$ and $y$, let $\mathcal{A}_{(x,y)}$ be the set of all level sets of $x$ and $y$, i.e. $\{X_{(i)} \mid i \in N\} \cup \{Y_{(i)} \mid i \in N\}$. Note that sets belonging to $\mathcal{A}_{(x,y)}$ are the only ones that appear in the objective function (1). This specificity can be exploited to simplify the regret optimization problem. Let us indeed consider now queries involving binary alternatives of type $1A0$, where $1A0$ represents a fictitious alternative with a top performance on all criteria in $A \subseteq N$ and a bottom performance on all others. More precisely, the DM may be asked to compare such alternatives to constant utility profiles of type $\Lambda = (\lambda, \ldots, \lambda)$. Note that by definition, $C_v(1A0) = v(A)$ and $C_v(\Lambda) = \lambda$ for any capacity $v$ and any set $A \subseteq N$. As a consequence, if the preference $1A0 \succsim \Lambda$ (resp. $1A0 \precsim \Lambda$) is observed, then Equation (5) gives the simple constraint $v(A) \geq \lambda$ (resp. $v(A) \leq \lambda$). Consequently, Equation (5) can be replaced by boundary constraints over decision variables; indeed, to ensure that the set-function $v$ is compatible with $\mathcal{P}$, it is sufficient to update the boundaries of an interval $[l_A, u_A]$ whenever a preference of type $1A0 \succsim \Lambda$ or $1A0 \precsim \Lambda$ is inserted in $\mathcal{P}$. Since $\Theta_{\mathcal{P}\cup\{1A0\succsim\Lambda\}}$ is the set of all capacities $v \in \Theta_\mathcal{P}$ that satisfy $v(A) \geq \lambda$, and keeping in mind that all capacities are monotonic by definition, then necessarily $v(B) \geq \lambda$ for all these capacities and all $B \supseteq A$ (i.e. $\Theta_{\mathcal{P}\cup\{1A0\succsim\Lambda\}} = \Theta_{\mathcal{P}\cup\{1B0\succsim\Lambda \mid B\supseteq A\}}$). Thus, if the preference $1A0 \succsim \Lambda$ is observed, then all preferences of type $1B0 \succsim \Lambda$ for $B \supseteq A$ can be inserted in $\mathcal{P}$. Similarly, if the preference $1A0 \precsim \Lambda$ is observed, then all preferences of type $1B0 \precsim \Lambda$ for $B \subseteq A$ can be inserted in $\mathcal{P}$.

Let $\mathcal{P}$ be a set of preferences statements obtained by inserting preferences of type $1A0 \succsim \Lambda$ (resp. $1A0 \precsim \Lambda$) and, for each of them, the preference $1B0 \succsim \Lambda$ for all $B \supseteq A$ (resp. $1B0 \precsim \Lambda$ for all $B \subseteq A$). Let $l_A, u_A$ be the resulting lower and upper bounds obtained for all $A \in 2^N$, then the following proposition holds:

**Proposition 1.** *Any function $v: \mathcal{A} \to [0,1]$ with $\mathcal{A} \subset 2^N$ such that*
*i) $v(A) \in [l_A, u_A]$ for all $A \in \mathcal{A}$, and*
*ii) $v(A) \leq v(B)$ for all $A, B \in \mathcal{A}$ such that $A \subset B$*
*can be completed into a capacity in $\Theta_\mathcal{P}$.*

*Proof.* By construction of $\mathcal{P}$ and all its corresponding intervals, it is sufficient to complete $v$ by setting first $v(A)$ to $l_A$ for all $A \in 2^N \backslash \mathcal{A}$ such that $|A| = 1$. Then, we iteratively set the value of $v(A)$ to $\max\{l_A, \max_{\{B\subset A \mid |B|=|A|-1\}} v(B)\}$ for all $A \in 2^N \backslash \mathcal{A}$ such that $v(B)$ is known for all $B \subset A$ such that $|B| = |A|-1$, so as to obtain a completely specified capacity. □

Thus, Proposition 1 enables one to conclude that all constraints given by Equation (4) involving $v_A$ for any $A \notin \mathcal{A}_{(x,y)}$ can be removed and so $\mathrm{PMR}(x,y;\Theta_\mathcal{P})$ can be computed by solving the following simpler linear program:

$$\begin{aligned}
\max_v \quad & C_v(y) - C_v(x) && (6)\\
s.t. \quad & v_{X_{(i+1)}} \leq v_{X_{(i)}} \quad \forall i \in [\![1; n-1]\!] && (7)\\
& v_{Y_{(i+1)}} \leq v_{Y_{(i)}} \quad \forall i \in [\![1; n-1]\!] && (8)\\
& v_{X_{(i)}} \leq v_{Y_{(j)}} \quad \forall i,j \in N \text{ s.t. } X_{(i)} \subset Y_{(j)} && (9)\\
& v_{Y_{(i)}} \leq v_{X_{(j)}} \quad \forall i,j \in N \text{ s.t. } Y_{(i)} \subset X_{(j)} && (10)\\
& l_{X_{(i)}} \leq v_{X_{(i)}} \leq u_{X_{(i)}} \quad \forall i \in N && (11)\\
& l_{Y_{(i)}} \leq v_{Y_{(i)}} \leq u_{Y_{(i)}} \quad \forall i \in N && (12)
\end{aligned}$$

Let $w_A$ denote the coefficient of the decision variable $v_A$ in the objective function (6), for any set of criteria $A \in \mathcal{A}_{(x,y)}$. Note that $w_A = -(x_{(i)} - x_{(i-1)}) \leq 0$ for all $A \in \{X_{(i)} \mid X_{(i)} \neq Y_{(i)}\}$, $w_A = y_{(i)} - y_{(i-1)} \geq 0$ for all $A \in \{Y_{(i)} \mid Y_{(i)} \neq X_{(i)}\}$ and $w_A = y_{(i)} - y_{(i-1)} - (x_{(i)} - x_{(i-1)})$ for all $A \in \{X_{(i)} \mid X_{(i)} = Y_{(i)}\}$. Since the objective function has to be maximized, we can deduce that $v_A$ will be as small as possible for all $A \in \{X_{(i)} \mid X_{(i)} \neq Y_{(i)}\}$ and as large as possible for all $A \in \{Y_{(i)} \mid Y_{(i)} \neq X_{(i)}\}$. Thus, none of the constraints (9) are required to find the optimum. Note also that some constraints given by Equation (10) are unnecessary. Indeed, if there exists $i, j \in N$ such that $Y_{(i)} \subset X_{(j)}$, then we also have $Y_{(i)} \subset X_{(k)}$ for all $k \in [\![1; j]\!]$, which creates redundant constraints added to Equation (7); thus, it is sufficient to impose $v_{Y_{(i)}} \leq v_{X_{(j)}}$ only if $Y_{(i)} \subset X_{(j)}$ and $Y_{(i)} \not\subseteq X_{(j+1)}$. However, if $Y_{(i+1)} \subseteq X_{(j)}$ is also satisfied, there is a redundancy with Equation (8). Finally, it is sufficient to impose $v_{Y_{(i)}} \leq v_{X_{(j)}}$ only if $Y_{(i)} \subset X_{(j)}$, $Y_{(i)} \not\subseteq X_{(j+1)}$ and $Y_{(i-1)} \not\subseteq X_{(j)}$. Thus, the number of monotonicity constraints is now below $3(n-1)$ and at most $2(n-1)$ variables are used (the elements of $\mathcal{A}_{(x,y)}$).

### 3.3 Efficient Optimization for PMR

Although the numbers of constraints and variables of the linear program defined in (6-12) are polynomial in the number of criteria, the computation time required by a state-of-the-art solver to obtain minimax regret increases significantly with the number of alternatives due to the quadratic number of PMR optimizations. We show now how the PMR-optimization problem can be solved efficiently by an iterative procedure, for any $x, y \in \mathcal{X}$ and any set of preference statements $\mathcal{P}$ of type $1A0 \succsim \Lambda$ or $1A0 \precsim \Lambda$, obtained by doing all the insertions described in the previous subsection.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ denote the constraint graph associated to Equations (7-10). $\mathcal{G}$ is defined as follows:

- $\mathcal{V}$ is the set of all decision variables in the linear program; thus, $\mathcal{V} = \{v_{X_{(i)}} \,|\, i \in N\} \cup \{v_{Y_{(i)}} \,|\, i \in N\}$.
- $\mathcal{A}$ is the set of arrows $(v_A, v_B)$ such that $v_A \geq v_B$ is given by Equations (7- 8) or Equation (10) without redundancy.

Note that the $2(n-1)$ constraints given by Equations (7 - 8) imply the existence of the two paths $V_x = (v_{X_{(1)}}, \ldots, v_{X_{(n)}})$ and $V_y = (v_{Y_{(1)}}, \ldots, v_{Y_{(n)}})$, which together include all the nodes in $\mathcal{V}$. Recall that $w_A$ denotes the coefficient of $v_A$ in the objective function (6). Let $V^-$ (resp. $V^+$) be the restricted sequence of $V_x$ (resp. $V_y$) to all nodes $v_A$ such that $w_A \leq 0$ (resp. $w_A \geq 0$ and $v_A \notin V^-$). As already noted before, $w_A \leq 0$ for all $A \in \{X_{(i)} \mid X_{(i)} \neq Y_{(i)}\}$ and $w_A \geq 0$ for all $A \in \{Y_{(i)} \mid X_{(i)} \neq Y_{(i)}\}$. Thus, $V^-$ and $V^+$ include together all the nodes in $\mathcal{V}$ and have no common node (see Figure 1 for illustration, where $x = (1, 0.8, 0.4, 0.5, 0.1)$ and $y = (0.8, 0.9, 0.6, 0.2, 0.4)$).

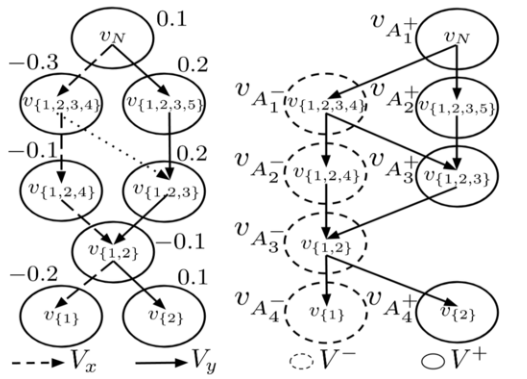


**Figure 1.** Illustration of $V^-$ and $V^+$ construction from $\mathcal{G}$.

Let $v_{A_i^+}$ (resp. $v_{A_i^-}$) denote the $i^{th}$ node of the sequence $V^+$ (resp. $V^-$). Note that $V^+$ (resp. $V^-$) includes all the variables that have a positive (resp. negative) impact on the objective function; hence we want to maximize (resp. minimize) the variables in $V^+$ (resp. $V^-$) so as to maximize the objective function. Thus, if there exists no arrow of type $(v_{A_i^-}, v_{A_j^+})$ in $\mathcal{A}$, then the optimum of the linear program can be easily obtained. Indeed, it is sufficient to set $v_A$ to its lower bound $l_A$ for all $v_A \in V^-$ and $v_A$ to its upper bound $u_A$ for all $v_A \in V^+$. Otherwise, for all arrow of type $(v_{A_i^-}, v_{A_j^+})$ in $\mathcal{A}$, we need to decide whether to assign the variable $v_{A_i^-}$ to the lower bound $l_{A_i^-}$ at the expense of constraining $v_{A_j^+}$ or to assign $v_{A_i^-}$ to an higher value. This can be done using Algorithm 1 given below, where $D^+(v_A)$ (resp. $D^-(v_A)$) denotes the restriction of $V^+$ (resp. $V^-$) to the descendants of $v_A$ in $\mathcal{G}$ and $D^+_j(v_A)$ denotes the $j^{th}$ element in sequence $D^+(v_A)$. In this algorithm, for each node $v_{A_i^-}$ we iteratively compute and compare quantities $w^+$ and $w^-$ (lines 7 and 14), where $w^+$ represents at step $j$ the overall weight of the $j$ first elements of $D^+(v_{A_i^-})$ and $w^-$ is the overall weight of their ancestors in $D^-(v_{A_i^-})$. In the example of Figure 1, for $i = 1$, we obtain $w^+ = w_{A_3^+} + w_{A_4^+} = 0.3$ and $w^- = w_{A_1^-} + w_{A_2^-} + w_{A_3^-} = -0.5$ at the end of the while loop. The correctness of Algorithm 1 can be proved using the following loop invariant: "For all $k < i$, $v(A_k^-)$ is equal to the capacity value of $A_k^-$ in the optimal solution and for all $v_A \in V^+$, $v(A)$ is equal to the maximum feasible capacity value of $A$ knowing the value of the latter nodes". Note that the condition of line 14 can be true only if the body of the while loop is executed at least once; in this case, $B$ is well defined.

**Algorithm 1:** Iterative optimization of PMR

**Input**: Two alternatives $x, y \in \mathcal{X}$
**Output**: $v$ defined on $\mathcal{A}_{(x,y)}$ achieving $\mathrm{PMR}(x, y; \Theta_{\mathcal{P}})$

```
1  Construction of V⁻ and V⁺ from x and y
2  foreach v_A ∈ V⁺ do v(A) ← u_A for i = 1 ... |V⁻| do
3      w⁻ ← w_{A_i^-}
4      w⁺ ← 0
5      j ← 1
6      while |w⁻| ≥ w⁺ and D_j^+(v_{A_i^-}) exists do
7          v_B ← D_j^+(v_{A_i^-})
8          if v(B) < l_{A_i^-} then break
9          w⁺ ← w⁺ + w_B
10         w⁻ ← w⁻ + Σ_{v_A ∈ D⁻(v_{A_i^-}): D_1^+(v_A) = v_B} w_A
11         j ← j + 1
12     end
13     if |w⁻| < w⁺ then v(A_i^-) ← v(B) else v(A_i^-) ← l_{A_i^-}
14     foreach v_A ∈ D⁺(v_{A_i^-}) do v(A) ← min{v(A), v(A_i^-)}
15 end
16 return v
```

## 4 AN INCREMENTAL ELICITATION METHOD FOR CHOQUET INTEGRALS

We introduce now a query strategy assuming the DM is only asked to compare binary alternatives to constant profiles. Our query selection strategy uses the WMMR criterion presented in Definition 4; thus, since the DM is only asked to compare a binary alternative 1A0 to a constant profile $\Lambda = (\lambda, \ldots, \lambda)$, an optimal query is defined by a pair $(A \subseteq N, \lambda \in [l_A; u_A])$ that brings the smallest minimax regret in the answer's worst-case scenario. In order to find such a pair, we have to determine, for all sets $A \subseteq N$, the value $\lambda_A^* \in [l_A, u_A]$ that minimizes the WMMR criterion; thus, an optimal query is defined by a pair $(A \subseteq N, \lambda_A^*)$ that minimizes the latter criterion.

Given a set $A \subseteq N$, determining $\lambda_A^*$ amounts to minimizing over $\lambda \in [l_A, u_A]$ the maximum between $\mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P} \cup \{1A0 \succsim \Lambda\}})$ and $\mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P} \cup \{1A0 \precsim \Lambda\}})$. Note that $\mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P} \cup \{1A0 \succsim \Lambda\}})$ and $\mathrm{MMR}(\mathcal{X}; \Theta_{\mathcal{P} \cup \{1A0 \precsim \Lambda\}})$ are two functions of $\lambda$ and that the former is a decreasing one while the latter is an increasing one. Similarly to what is observed for utility functions over consequences [27], these two functions necessarily intersect since they have the same maximum (i.e. $\mathrm{MMR}(\mathcal{X};\Theta_{\mathcal{P}})$). This intersection gives the value of $\lambda_A^*$ and can easily be computed by a bisection algorithm relying on the relative positions of the two curves, observed at two distinct points. However, it may happen that the WMMR value of the optimal query is equal to $\mathrm{MMR}(\mathcal{X};\Theta_{\mathcal{P}})$, which means that the latter question will not necessarily induce a regret reduction. In such cases, our procedure chooses a set $A \subseteq N$ that minimizes, for $\lambda = (l_A + u_A)/2$, the expected value of *minimax regret* over the two possible answers with an uniform distribution hypothesis over $[l_A, u_A]$.

Note that determination of the next query implies to select $A$ within the $2^n - 2$ possible proper subsets of $N$, a number which increases significantly with the number of criteria. To make this query selection step more efficient, we propose, as a heuristic, to focus on sets directly involved in the computation of $\mathrm{PMR}(x^*, y^*; \Theta_{\mathcal{P}})$, where $x^*$ is an optimal solution for the MMR criterion knowing $\mathcal{P}$, and $y^*$ is one of the adversary's choices. These sets are those in $\mathcal{A}_{(x^*,y^*)} = \{X^*_{(i)} \mid i \in N\} \cup \{Y^*_{(i)} \mid i \in N\}$, where $X^*_{(i)}$ and $Y^*_{(i)}$ respectively denote the $i^{th}$ level set of $x^*$ and $y^*$. Thus, the heuristic will further constrain parameters involved in the computation of the pairwise max regret of $x^*$ with respect to $y^*$ and possibly reduce the minimax regret. According to this heuristic, at most $2n - 2$ sets are investigated (elements of $\mathcal{A}_{(x^*,y^*)}$) instead of exactly $2^n - 2$.

## 5 EXPERIMENTS

In this section, we report a number of numerical tests. The first ones aim at comparing the computation time of minimax regret calculation when using either a solver (LP) to optimize (6-12) or the iterative optimization (IO) algorithm presented in Section 3.3. To do so, we consider two datasets ("Knapsack5" and "Knapsack10") consisting of the Pareto set of two multi-objective knapsack problems ($n = 5$ and $n = 10$), restricted to one thousand of alternatives. Results have been obtained by averaging over 50 runs. In Table 1, we can see that IO is significantly faster than LP (about five orders of magnitude). In fact, IO allows our incremental procedure to ask fifty queries in a few minutes for about a thousand of alternatives, while LP takes about six hours for ten times less alternatives[3].

**Table 1.** Comparaison of minimax regret computation time in seconds.

| Dataset | Query | LP | IO |
|---|---|---|---|
| Knapsack5 | 0 | 34.594 | 0.005 |
| Knapsack5 | 10 | 27.583 | 0.005 |
| Knapsack5 | 20 | 28.895 | 0.005 |
| Knapsack10 | 0 | 31.120 | 0.017 |
| Knapsack10 | 10 | 15.197 | 0.007 |
| Knapsack10 | 20 | 13.016 | 0.006 |

The second experiments aim at evaluating the efficiency of our query strategy. Starting from an empty set of preferences $\mathcal{P}$, simulated users answer to queries according to a Choquet Integral drawn at random. We implement the elicitation procedure introduced in Section 4 and compute both the minimax regret and real regret (obtained thanks to the user simulated utility model) at each iteration step. Results have been obtained by averaging over 200 runs and are given in Figure 2. We can see that the minimax regret reduces reasonably quickly as the number of preference queries increases and that the real regret is much smaller (a fact that has already been observed a number of times in regret-based elicitation [27, 25, 3]).

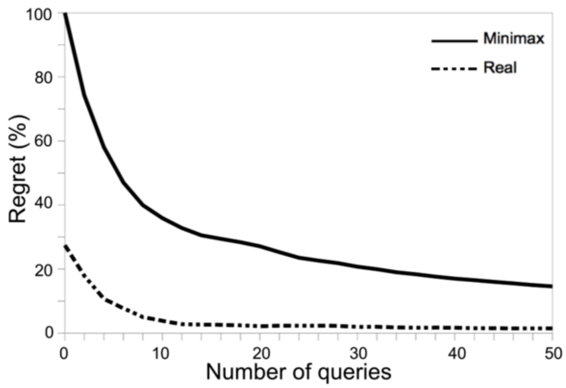


**Figure 2.** Incremental elicitation procedure (Knapsack10).

Now, we want to compare our incremental preference elicitation procedure based on the Choquet model to the standard elicitation method based on a linear aggregation function and the Current Solution Strategy (CSS) as presented in [3] (based on the comparison of $x^*$ and $y^*$ at each step, see subsection 2.2). To do so, datasets of 100 alternatives evaluated on 10 criteria and characterized by a set of performance vectors $\mathcal{X}^a$ are randomly generated. They are constructed in such a way that $\sum_{i=1}^n x_i^a = 1$ for all $x \in \mathcal{X}^a$, where $a \in \{0.5, 1, 2\}$ so as to obtain different types of Pareto sets (controlling the proportion and the location of non-supported[4] Pareto-optimal solutions). We only report results with simulated users answering to queries according to a concave multiattribute utility function so as to model a preference in favour of "well-balanced solutions". In the general case (including linear utility model), after at most 20 queries, both procedures recommend an alternative with a real regret under ten percent of the maximum real regret in the dataset, on average. Results are obtained by averaging over 100 runs.

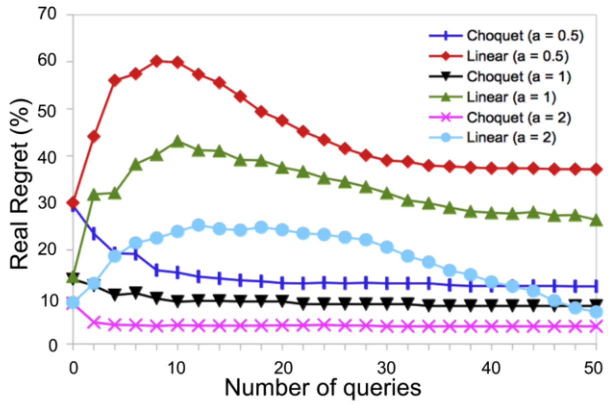


**Figure 3.** Comparison of incremental elicitation procedures.

In Figure 3, we can see that the real regret, at any step of the elicitation process, is smaller with our procedure based on the Choquet model than with the procedure based on the linear model for any dataset $\mathcal{X}^a$. Indeed, in most cases, there exists well-balanced Pareto-optimal alternatives (presumably very attractive for the DM) that can-

[3] Linear optimizations are done using the Gurobi library of Java.

[4] Solutions that do not belong to the boundary of the convex hull of $\mathcal{X}^a$.

not be obtained by optimizing a weighted sum of criterion values and consequently, reducing the space of possible weights cannot lead to the recommendation of such alternatives. On the contrary, various of these can be attained by maximizing a concave Choquet integral.

## 6 DISCUSSION AND CONCLUSION

In this paper we discussed the problem of interactively eliciting a Choquet capacity using a minimax regret approach. Technical difficulties are related to the number of parameters needed to characterize a given capacity and the number of constraints required to characterize the space of admissible capacity functions. We showed how, assuming that preferences are stated in a particular form, minimax regret optimization can be performed efficiently (in particular we presented both a linear programming formulation and an even faster algorithm maintaining lower and upper bounds). We presented experimental results validating both the computational efficiency of our approach in large problem instances and the quality of recommendations elaborated through our incremental process.

Our work differentiates from previous work on Choquet integrals in the focus on incremental elicitation; minimax regret provides robust recommendations. Notably, Ah-Pine et al. [1] assess a feasible capacity for a Choquet integral given some preferential information that maximize the margin of the induced constraints (in a fashion similar to SVM classifiers). This kind of "pointwise" estimation ignore however the specificity of the available items. Moreover, it does not directly provide a natural strategy for choosing the query to ask in an incremental elicitation setting.

A first direct continuation of this work is to extend the elicitation procedure for set recommendation. The approach we have proposed in this paper extends naturally to sets but is computationally more demanding. Possible future works include further experimental work and validation with real users. An interesting direction of research is incremental elicitation of the capacity of a Choquet integral using a Bayesian approach (following [26]).

## ACKNOWLEDGEMENTS

Work supported by the French National Research Agency through the Idex Sorbonne Universités under grant ANR-11-IDEX-0004-02.

## REFERENCES

[1] J. Ah-Pine, B. Mayag, and A. Rolland, 'Identification of a 2-additive bi-capacity by using mathematical programming', in *Algorithmic Decision Theory*, 15–29, Springer, (2013).

[2] C. Boutilier, 'A POMDP Formulation of Preference Elicitation Problems', in *Proc. of AAAI-02*, pp. 239–246, Edmonton, (2002).

[3] C. Boutilier, R. Patrascu, P. Poupart, and D. Schuurmans, 'Constraint-based Optimization and Utility Elicitation using the Minimax Decision Criterion', *Artifical Intelligence*, **170**(8–9), 686–713, (2006).

[4] D. Braziunas, *Decision-theoretic elicitation of generalized additive utilities*, Ph.D. dissertation, University of Toronto, 2011.

[5] D. Braziunas and C. Boutilier, 'Assessing regret-based preference elicitation with the utpref recommendation system', in *Proceedings 11th ACM Conference on Electronic Commerce (EC-2010)*, pp. 219–228, (2010).

[6] U. Chajewska, D. Koller, and R. Parr, 'Making Rational Decisions Using Adaptive Utility Elicitation', in *Proc. of AAAI-2000*, pp. 363–369, Austin, TX, (2000).

[7] L. Galand, J. Lesca, and P. Perny, 'Dominance rules for the Choquet integral in multiobjective dynamic programming', in *proc. of IJCAI'13*, pp. 538–544, (2013).

[8] L. Galand, P. Perny, and O. Spanjaard, 'Choquet-based optimisation in multiobjective shortest path and spanning tree problems', *European Journal of Operational Research*, **204**(2), 303–315, (2010).

[9] M. Grabisch and C. Labreuche, 'A decade of application of the Choquet and Sugeno integrals in multi-criteria decision aid', *Annals of Operations Research*, **175**(1), 247–286, (2010).

[10] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap, *Aggregation Functions*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, New-York, 2009.

[11] M. Grabisch, H.T. Nguyen, and E.A. Walker, *Fundamentals of Uncertainty Calculi, with Applications*, Encyclopedia of Mathematics and its Applications, Kluwer Academic Publishers, 1995.

[12] F. Huédé, M. Grabisch, C. Labreuche, and P. Savéant, 'Integration and propagation of a multi-criteria decision making model in constraint programming', *Journal of Heuristics*, **12**(4-5), 329–346, (2006).

[13] E. Hüllermeier and A. Fallah Tehrani, 'Efficient learning of classifiers based on the 2-additive Choquet integral', in *Computational Intelligence in Intelligent Data Analysis Studies in Computational Intelligence Volume*, volume 445, pp. 17–29, (2013).

[14] C. C. White III, A. P. Sage, and S. Dozono, 'A model of multiattribute decisionmaking and trade-off weight determination under uncertainty', *IEEE Transactions on Systems, Man, and Cybernetics*, **14**(2), 223–229, (1984).

[15] P. Meyer J.-L. Marichal and M. Roubens, 'Sorting multi-attribute alternatives: the TOMASO method', *Computers & Operations Research*, **32**, 861–877, (2005).

[16] P. Kouvelis and G. Yu, *Robust Discrete Optimization and Its Applications*, Kluwer, Dordrecht, 1997.

[17] J. Lesca, M. Minoux, and P. Perny, 'Compact versus Noncompact LP Formulations for minimizing Convex Choquet Integrals', *Discrete Applied Mathematics*, (161), 184–199, (2013).

[18] J.-L. Marichal and M. Roubens, 'Determination of weights of interacting criteria from a reference set', *European Journal of Operational Research*, **124**(3), 641–650, (2000).

[19] P. Meyer and M. Roubens, 'On the use of the Choquet integral with fuzzy numbers in multiple criteria decision support', *Fuzzy Sets and Systems*, **157**(7), 927–938, (2006).

[20] A. Salo and R. P. Hämäläinen, 'Preference ratios in multiattribute evaluation (PRIME)–elicitation and decision procedures under incomplete information', *IEEE Trans. on Systems, Man and Cybernetics*, **31**(6), 533–545, (2001).

[21] L. J. Savage, *The Foundations of Statistics*, Wiley, New York, 1954.

[22] A. F. Tehrani, W. Cheng, K. Dembczynski, and E. Hüllermeier, 'Learning monotone nonlinear models using the choquet integral', *Machine Learning*, **89**(1-2), 183–211, (2012).

[23] M. Timonin, 'Maximization of the Choquet integral over a convex set and its application to resource allocation problems', *Annals of Operations Research*, **196**, 543–579, (2012).

[24] V. Torra, 'The weighted OWA operator', *International Journal of Intelligent Systems*, **12**, 153–166, (1997).

[25] P. Viappiani and C. Boutilier, 'Regret-based optimal recommendation sets in conversational recommender systems', in *Proceedings of the 3rd ACM Conference on Recommender Systems (RecSys09)*, pp. 101–108, New York, (2009).

[26] P. Viappiani and C. Boutilier, 'Optimal bayesian recommendation sets and myopically optimal choice query sets', in *Advances in Neural Information Processing Systems 23 (NIPS)*, pp. 2352–2360, (2010).

[27] T. Wang and C. Boutilier, 'Incremental Utility Elicitation with the Minimax Regret Decision Criterion', in *IJCAI-03*, pp. 309–316, (2003).

[28] M.E. Yaari, 'The dual theory of choice under risk', *Econometrica*, **55**, 95–115, (1987).

[29] R.R. Yager, 'On Ordered Weighted Averaging aggregation operators in multicriteria decision making', *IEEE Trans. Systems, Man and Cybern.*, **18**, 183–190, (1988).