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A Centrality Measure for Networks With Community Structure Based on a Generalization of the Owen Value

Piotr L. Szczepański[1] and **Tomasz P. Michalak**[2,3] and **Michael Wooldridge**[2]

Abstract. There is currently much interest in the problem of measuring the *centrality* of nodes in networks/graphs; such measures have a range of applications, from social network analysis, to chemistry and biology. In this paper we propose the first measure of node centrality that takes into account the *community structure* of the underlying network. Our measure builds upon the recent literature on game-theoretic centralities, where solution concepts from cooperative game theory are used to reason about importance of nodes in the network. To allow for flexible modelling of community structures, we propose a generalization of the Owen value—a well-known solution concept from cooperative game theory to study games with *a priori*-given unions of players. As a result we obtain the first measure of centrality that accounts for both the value of an individual node's relationships within the network and the quality of the community this node belongs to.

1 INTRODUCTION

Real-world networks frequently have highly complex structures. They can often be characterised by properties such as heavy-tailed degree distributions, clustering, the small-world property, etc. Another important characteristics that many real-life networks have in common is their *community structure* [15, 21]. Communities are usually composed of nodes that are more densely connected internally than with other nodes in the network. For instance, the teachers from a particular secondary school may form a community within the social network of all teachers in the city. Similarly, trade links among the European Union countries are usually more intense than their links with the rest of the world. In addition, certain communities may be considered to be stronger than others. Secondary schools may vary in reputation, and some trade blocks may be more important to the global economy than others.

The importance of a community is usually increased when a new, powerful individual joins it. Conversely, membership in a strong community may boost the importance of an otherwise weak individual. Quantifying this latter effect is the primary goal of this paper. In other words, we are concerned with the problem of analysing the importance (the *centrality*) of individual nodes given the underlying community structure of the network.

Centrality analysis is an important research issue in various domains, including social networks, biology and computer science [14, 10]. Four widely-known *centrality measures* are degree, closeness, betweenness and eigenvector centralities [14, 7]. On top of these well-known standard measures, many other — often more sophisticated — approaches have been considered in the literature. Recently, various methods for the analysis of cooperative games have been advocated as measures of centrality [16, 10]. The key idea behind this approach is to consider groups (or coalitions) of nodes instead of only considering individual nodes. By doing so this approach accounts for potential synergies between groups of nodes that become visible only if the value of nodes are analysed jointly [18]. Next, given all potential groups of nodes, game-theoretic solution concepts can be used to reason about players (i.e., nodes) in such a coalitional game.

One interesting advantage of game-theoretic centralities is their flexibility. In particular, there are very many ways in which a coalitional game can be defined over any given network. Furthermore, there are many well-studied and readily-available solution concepts—such as the *Shapley value* [25] and the *Banzhaf index* [6]—with which to analyse the network from different angles. In this paper, we use the flexibility offered by the game-theoretic approach to construct the first centrality measure in the literature that is able to account for complex community structures in networks. To this end, we model the community structure as the *a priori* given coalition structure of a cooperative game. By doing so, we are able to build a centrality metric by generalizing the Owen value [23]—a well-known solution concept for cooperative games in which players are partitioned into pre-defined groups.

In our approach, the computation of a node's power is a two-step process. First, we compute the importance of the community (if any) that this node belongs to. Next, we compute the power of the given node within this community. Our generalization of the Owen value, which we call *coalitional semivalues*, is a much broader solution concept. In fact, coalitional semivalues encompass the Owen value as well as all other solution concepts in the literature that were developed for games with an *a priori* defined coalition structure of players: the Owen-Banzhaf value [24], the symmetric coalitional Banzhaf value, and p-binomial semivalues [8].

Unfortunately, game-theoretic centrality measures are often computationally complex. In particular, the Shapley value is embedded in the definition of the Owen value, and is known to be NP-hard for many representations of games [9]. This negative result also extends to various coalitional games defined over networks [4, 5] and game-theoretic centralities in particular [19, 26]. Although, in general, the new centrality introduced in this paper is #P-complete (and hence NP-hard), we are able to give a polynomial algorithm to compute it for problem instances where the value of any group of nodes is determined by their degree centrality [14]. We verify the practical aspects of our algorithm on a large citation network that contains more than 2 million nodes and links. Our experiments compare three different degree centralities: group degree centrality [12], the Shapley value-based degree centrality [18], and our new centrality. We show that,

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[1] Warsaw University of Technology, Poland
[2] University of Oxford, UK
[3] University of Warsaw, Poland

unlike others, our new centrality produces a ranking in which the power of the top nodes significantly differs, depending on the power of the communities that these nodes belong to.

2 PRELIMINARIES

A *cooperative game in characteristic function form* (hereafter just "game") consists of a set of players $N = \{1, 2, \dots, n\}$ and a *characteristic function*[4] $\nu : 2^N \to \mathbb{R}$. The characteristic function assigns to each *coalition* $C \subseteq N$ of players a real value (or payoff) indicating its performance. We assume that $\nu(\emptyset) = 0$. A *coalition structure*, denoted by CS, is a partition of N into disjoint coalitions. Formally, $N = \bigcup_{C_i \in CS} C_i$ and $\forall C_i, C_j \in CS$ if $i \neq j$ we have $C_i \cap C_j = \emptyset$.

One of the fundamental problems in cooperative game theory is how to evaluate the importance or contribution of players in the coalitional game. *Semivalues* represent an important class of solutions to this problem [11]. To define semivalues, let us denote by $\text{MC}(C, v)$ the marginal contribution of the player i to the coalition C, i.e.,

$$\text{MC}(C, i) = \nu(C \cup \{i\}) - \nu(C).$$

Let $\beta : \{0, 1, \dots, |N| - 1\} \to [0, 1]$ be a discrete probability distribution. Intuitively, $\beta(k)$ will be the probability that a coalition of size k is drawn from the set of all possible coalitions. Given the function β, the semivalue $\phi_v(\nu)$ of a player i in the cooperative game ν is:

$$\phi_i(\nu) = \sum_{0 \leq k < |N|} \beta(k) \mathbb{E}_{C^k}[\text{MC}(C^k, i)], \tag{1}$$

where C^k is the random variable that corresponds to a coalition being drawn with uniform probability from the set of all coalitions of size k in the set of players $N \setminus \{i\}$. $\mathbb{E}_{C^k}[\cdot]$ is the expected value operator for C^k.

A prominent semivalue was proposed by Shapley [25]. The *Shapley value* is proven to be the unique payoff division scheme of the grand coalition satisfying four, often desirable, properties: Efficiency, Symmetry, Null player and Additivity [25]. Another prominent example of a semivalue with applications in measuring the power of players in cooperative games is the *Banzhaf index* of power [6]. This latter semivalue satisfies Symmetry, Null player and Additivity but not Efficiency. The Shapley value and the Banzhaf index are defined by their respective β-functions $\beta^{Shapley}$ and $\beta^{Banzhaf}$:

$$\beta^{Shapley}(i) = \frac{1}{|N|} \quad \text{and} \quad \beta^{Banzhaf}(i) = \frac{\binom{|N|-1}{i}}{2^{|N|-1}}.$$

A *network* is defined as a graph $G = (V, E)$, where V is a set of nodes and E is a set of edges containing unordered pairs (v, u) of nodes $v, u \in V$. We define the set of *neighbours* of a node v by $N(v) = \{u : (u, v) \in E\}$ and for any set of nodes, $S \subseteq V$, we define $N(S) = \bigcup_{v \in S} N(v) \setminus S$, respectively. The *degree* of a node v is simply the number if its neighbours $deg(v) = |N(v)|$.

Nodes within the network G can form communities $C_G \subseteq V$. The *community structure*, denoted by CS_G, is an exhaustive partition of V into disjoint communities. Formally, $V = \bigcup_{C_G \in CS_G} C_G$ and $\forall C'_G, C''_G \in CS_G$ we have $C'_G \cap C''_G = \emptyset$. These properties also satisfy the requirements for coalition structures. In other words, if we view the network as a population of interconnected agents playing a coalition game, then the community structure within the network can be straightforwardly interpreted as the *a priori* given coalition structure of agents. More formally, let us define a cooperative game played on graph $G = (V, E)$ by the pair (V, ν_G), where $\nu_G : 2^V \to \mathbb{R}$ is a characteristic function that assigns to each community $C_G \subseteq V$ a real value indicating its importance, with $\nu_G(\emptyset) = 0$. Since communities in this model are coalitions and community structures are coalition structures, we unify the notation and refer to them by C and CS, respectively.

Semivalues evaluate the contribution of a player to a coalition game. The key idea behind the game-theoretic network centrality (in our case semivalue-based centrality), is to use these solution concepts to evaluate the contribution of a node to the network, given the coalitional game (V, ν_G). Formally:

Definition 1 *The semivalue-based network centrality is a triple (G, ν_G, ϕ), where the value of each node $v \in V$ is given by $\phi_v(\nu_G)$.*

3 COALITIONAL SEMIVALUES & A NEW CENTRALITY MEASURE

Coalitional games can be analysed from both the *ex ante* and *ex post* perspectives. In the *ex ante* perspective, it is not known which coalition will actually form. The semivalues defined in formula (1) are *ex ante* since they consider the sum of the marginal contributions of a player to all possible coalitions without any additional assumptions.

Another approach is to consider a coalitional game from the *ex post* perspective. In this case it is already known which coalitions form by the end of the game. In other words, it is known how the agents have partitioned themselves into a coalition structure. This *ex post* perspective is especially appealing for our model where networks are or can be divided into communities.

In the remainder of this section we will propose an answer to the following question: how should we evaluate the importance of individual nodes within a network so that the relative importance of their communities is taken into account? To this end, we will consider how to adapt game-theoretic centralities so that they allow for an *a priori* given community structure of the social network.

The most popular extension of the Shapley value to "*ex post*"-like situations was proposed by Owen [23]. We will now define it using terminology appropriate to game-theoretic network centrality. To this end, let us first introduce the concept of the *quotient game* ν^Q. Given the coalitional game (V, ν_G) and community structure $CS = \{C_1, C_2, \dots, C_m\}$, we define a new coalitional game, where the communities are considered to be individual players:

$$\nu_G^Q(R) = \nu_G\left(\bigcup_{r \in R} C_r\right) \text{ for all } R \subseteq M,$$

where the set $M = \{1, 2, \dots, m\}$ represents coalitions' numbers. Note that $\bigcup_{r \in R} C_r$ is a *coalition of communities*. We will denote such coalition by Q_R.

We are now ready to define the Owen value. Given (V, ν_G) and $CS = \{C_1, C_2, \dots, C_m\}$, the share of the grand coalition's payoff, $\nu_G(V)$, given to player $i \in C_j \in CS$ according to the Owen value is given by:

$$OV_i(\nu_G, CS) = \sum_{R \subseteq M \setminus \{j\}} \frac{1}{|M|\binom{|M|-1}{|R|}} \sum_{C \subseteq C_j \setminus \{i\}} \frac{1}{|C_j|\binom{|C_j|-1}{|C|}} \text{MC}(Q_R \cup C, i).$$

The Owen value is the unique division scheme that satisfies the following five often desirable properties: Efficiency, Symmetry, Null player, Additivity, and Component Symmetry [23].

Let us examine the above formula more closely. The computation of the Owen value can be thought of as a two-step process. In the first

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[4] A cooperative game is a pair (N, ν), but we will usually refer to it simply as ν.

step, communities play the game (M, ν_G^Q) between themselves and receive their Shapley values. In the second step, the values of these communities are, in turn, divided amongst their members according to the Shapley value of the members.

In this paper we introduce a generalization of the Owen value, where more general division schemes—semivalues—are used as opposed to the Shapley value. Specifically, we combine formula (1) with the formula for the Owen value and propose *coalitional semivalues*, which we define as follows:

$$\phi_i(\nu_G, CS) = \sum_{0 \leq k < |M|} \beta(k) \sum_{0 \leq l < |C_j|} \alpha_j(l) \ \mathbb{E}_{T^k, C^l}[\text{MC}(Q_{T^k} \cup C^l, i)], \quad (2)$$

where T^k is a random set of size k drawn uniformly from the set $M \setminus \{j\}$, and C^l a the random set of size l drawn uniformly from the set $C_j \setminus \{i\}$. The function $\beta : \{0, 1, \ldots, |M| - 1\} \to [0, 1]$ is a function such that $\sum_{k=0}^{|M|-1} \beta(k) = 1$. $\{\alpha_j\}_{j \in \{1,\ldots,|M|\}}$ is a family of functions such that $\alpha_j : \{0, 1, \ldots, |C_j| - 1\} \to [0, 1]$ and $\sum_{k=0}^{|C_j|-1} \alpha_j(k) = 1$.

Intuitively, β is a probability distribution used to compute $\phi_j(M, \nu_G^Q)$, and α_j is the probability distribution used to evaluate the players inside a coalition $\phi_i(C_j, \nu)$. Importantly, as shown in Table 1, by adopting various probability distributions, we can obtain the Owen value [23], as well as all of its modifications proposed to date in the literature: *Owen-Banzhaf value* [24], *symmetric coalitional Banzhaf value* [2], and *symmetric coalitional p-binomial semivalues* [8].[5]

Table 1. Values of α and β for the Owen value and its various extensions.

Solution name	$\beta(k)$	$\alpha_j(l)$
Owen value [23]	$\frac{1}{\lvert M\rvert}$	$\frac{1}{\lvert C_j\rvert}$
Owen-Banzhaf value [24]	$\frac{\binom{\lvert M\rvert-1}{k}}{2^{\lvert M\rvert-1}}$	$\frac{\binom{\lvert C_j\rvert-1}{l}}{2^{\lvert C_j\rvert-1}}$
symmetric coalitional Banzhaf value [2]	$\frac{\binom{\lvert M\rvert-1}{k}}{2^{\lvert M\rvert-1}}$	$\frac{1}{\lvert C_j\rvert}$
symmetric coalitional p-binomial semivalue [8]	$p^k(1-p)^{\lvert M\rvert-1-k}$ $p \in [0,1]$	$\frac{1}{\lvert C_j\rvert}$

Let us now introduce the game-theoretic network centrality measure based on coalitional semivalues:

Definition 2 *The game-theoretic network centrality for the graph G with community structure CS is a quadruple (G, CS, ν_G, ϕ), where the value of each node $v \in V$ is given by $\phi_v(\nu_G, CS)$.*

This is the first centrality measure that evaluates nodes by taking into account the community structure of the network. In the next section, we will consider various properties of this new measure.

4 PROPERTIES

The aim of this section is to translate the properties of various instances of coalitional semivalues into the properties of the resulting centrality measure. The first three properties are derived from Null Player, Additivity and Symmetry, respectively.

Property 1 *If a node makes no contribution to any community then its value is zero:* $\forall_{C \subseteq V \setminus \{v\}} \text{MC}(C, v) = 0 \implies \phi_v(\nu_G, CS) = 0$.

Property 2 *If two group centralities are combined into one* $\nu_G = \nu'_G + \nu''_G$ *then* $\phi_v(\nu_G, CS) = \phi_v(\nu'_G, CS) + \phi_v(\nu''_G, CS)$.

Property 3 *If two nodes from the same community* $v, u \in C_j$ *contribute the same value to all possible communities then they are equally important:* $\forall_{C \subseteq V \setminus \{v,u\}} \text{MC}(C, v) = \text{MC}(C, u) \implies \phi_v(\nu_G, CS) = \phi_u(\nu_G, CS)$.

The next property involves the Quotient game:

Property 4 *The power of a community is the aggregation of the power of nodes comprising this community. Formally:* $\phi_j(M, \nu_G^Q) = \sum_{v \in C_j} \phi_v(\nu_G, CS)$.

All the solutions, where the power inside the communities is computed using the Shapley value (due to the Efficiency), have the above property. In the same spirit, it can required that all the power of the whole network $\nu_G(V)$ is distributed among the nodes:

Property 5 *The value of the whole network $\nu(V)$ is the aggregation of the power of nodes comprising this network:* $\nu_G(V) = \sum_{v \in V} \phi_v(\nu_G, CS)$.

Our final property is the translation of Component Symmetry. If we define the marginal contribution of the coalition C to the set of nodes Q_T as $\text{MC}(Q_T, C) = \nu(Q_T \cup C) - \nu(Q_T)$, we get:

Property 6 *If two communities contribute the same value to all possible groups of communities then their evaluation is the same:* $\forall_{T \subseteq M \setminus \{i,j\}} \text{MC}(Q_T, C_i) = \text{MC}(Q_T, C_j) \implies \phi_i(M, \nu_G^Q) = \phi_j(M, \nu_G^Q)$.

Table 2 summarizes properties of the coalitional semivalues.

Table 2. The properties of coalitional semivalue and its various instances.

Solution name	$P1$	$P2$	$P3$	$P4$	$P5$	$P6$
coalitional semivalue [this paper]	✓	✓	✓	×	×	×
Owen value [23]	✓	✓	✓	✓	✓	✓
Owen-Banzhaf value [24]	✓	✓	✓	×	×	×
symmetric coalitional Banzhaf value [2]	✓	✓	✓	✓	×	✓
symmetric coalitional p-binomial semivalue [8]	✓	✓	✓	✓	×	✓

5 COMPUTATIONAL ANALYSIS

For many succinctly represented coalitional games, computing the Shapley value is NP-hard (in fact, it is often #P-complete [9, 19]). Naive algorithms to compute the Shapley value (exhaustively computing the average marginal contribution over all orderings of players) have exponential running time. Given this, there are two possible research directions. Firstly, efficient approximate algorithms can be developed. Secondly, classes of centralities, that have real-life applications and can be computed in polynomial time, can be defined. In this paper, we take the latter approach and propose a polynomial time algorithm for computing coalitional semivalue-based centralities, where the characteristic function—the value of any group of nodes—is based on their degree. Thus, our method is build upon degree centrality — an important method of evaluating nodes in social networks analysis [14, 12, 18].

5.1 Weighted degree centralities

In this subsection we define a class of cooperative games such that a node's value is based on its degree. The *weighted group degree centrality* of a community C in graph G is defined as follows:

$$\nu_G^D(C) = \sum_{v \in N(C)} f(v), \quad (3)$$

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[5] We refrain from the axiomatic characterization of the new solution concept as being out of the scope of this paper.

where $N(C)$ is the set of neighbours of C, and f is a parameter that is a polynomially computable function.

Definition 3 *A game-theoretic network degree centrality for the graph G with the community structure CS is a quadruple $(G, CS, \nu_G^D(C), \phi)$, where the value of each node $v \in V$ is given by $\phi_v(\nu_G^D, CS)$.*

In the next section we will look more closely at the marginal contributions of nodes in order to effectively compute their expected value. This, in turn, will let us compute coalitional semivalues based on weighted group degree centrality in polynomial time.

5.2 The marginal contribution analysis

In this subsection we lay the groundwork for the efficient algorithm that will compute coalitional semivalues for weighted group degree centrality in polynomial time. To this end, we will use equation (2). For a given node, $v \in C_j \in CS$, the focus will be on computing the expected value of its marginal contribution: $\mathbb{E}_{T^k,C^l}[\mathrm{MC}(Q_{T^k} \cup C^l, v)]$.

We must consider the value of the expected marginal contribution of a node v to the set $Q_{T^k} \cup C^l$, where T^k is a random set of size k, and C^l is a random set of size l. Both sets are drawn uniformly from the sets $M \setminus \{j\}$ and $C_j \setminus \{v\}$, respectively. We will construct the effective algorithm in two steps. First, we will decide under what conditions v makes a contribution to the set $Q_{T^k} \cup C^l$. Second, we will use a combinatorial argument to compute this contribution for each of the cases distinguished in the first step.

Before we start, we need to introduce the following notation: for the node $v \in C_j \in CS$ we define the set of adjacent communities as $N_{CS}(v) = \{C_i \in CS \setminus C_j \mid C_i \cap N(v) \neq \emptyset\}$, inter-community degree as $deg_{CS}(v) = |N_{CS}(v)|$, the set of neighbours within a community as $N_j(v) = N(v) \cap C_j$, and intra-community degree as $deg_j(v) = |N_j(v)|$.

Theorem 1 *The game-theoretic network degree centrality for graph G with community structure CS for the node $v \in V$ can be computed in time polynomial in $|V|$.*

Proof: In our proof we will use concepts from probability theory. Thus, we will first define the probability space, which is a triple $(\Omega, \mathcal{F}, P)$, where Ω is a sample space containing sets $\{Q_{T^k} \cup C^l\}$, $T^k \subseteq 2^{|M|-1}$ is such that $|T^k| = k$ and $C^l \subseteq 2^{|C_j|-1}$ is such that $|C^l| = l$. The important observation is that $|\Omega| = \binom{|M|-1}{k}\binom{|C_j|-1}{l}$. In our model, $\mathcal{F}$ is the set of elementary events ($\mathcal{F} = \Omega$), and $P : \mathcal{F} \to [0, 1]$ is a probability distribution function such that for each event $A \in \mathcal{F}$ we have $P(A) = \frac{1}{|\Omega|}$.

There are two types of marginal contribution that a node can make. For the first, let us consider the marginal contribution of a single vertex v to the random set $Q_{T^k} \cup C^l$. When v joins a coalition C, it can contribute to its value with the help of any vertex $u \in N(v)$ if and only if u is not in C and u is not already directly connected to C. Let us introduce the Bernoulli random variable $B^{[1]}_{v,u,k,l}$, which will indicate whether the vertex v makes a contribution through vertex u to the random set $Q_{T^k} \cup C^l$. Equation (3) tells us that this contribution will be $f(u)$. Thus, we have:

$$\mathbb{E}[f(u)B^{[1]}_{v,u,k,l}] = f(u)P[(N(u) \cup \{u\}) \cap (Q_{T^k} \cup C^l) = \emptyset],$$

where $P[\cdot]$ denotes probability, and $\mathbb{E}[\cdot]$ denotes expected value.

The second type of contribution takes place when vertex v joins a coalition C and takes away the value $f(v)$. Such a contribution happens when vertex v is directly connected to the coalition C. In particular, weighted group degree centrality ν_G^D assumes that the value of a set of vertices depends only on nodes directly connected to this set, ignoring nodes already inside it. Therefore, when the node v becomes a member of C and it is not any more directly connected with it, the value of C is reduced by $f(v)$. Let us introduce the Bernoulli random variable $B^{[2]}_{v,u,k,l}$, which will indicate whether vertex v makes a contribution through itself to the random set $Q_{T^k} \cup C^l$. More formally, we have:

$$\mathbb{E}[-f(v)B^{[2]}_{v,k,l}] = -f(v)P[v \in N(Q_{T^k} \cup C^l)].$$

Now, we will move on to the second step of the proof and use a combinatorial argument to compute $P^{[1]} = P[(N(u) \cup \{u\}) \cap (Q_{T^k} \cup C^l) = \emptyset]$ and $P^{[2]} = P[v \in N(Q_{T^k} \cup C^l)]$.

Recall that there are exactly $\binom{|M|-1}{k}\binom{|C_j|-1}{l}$ sets $Q_{T^k} \cup C^l$. This is the size of the sample space. With this in mind, the probability $P^{[1]}$ for $u \in N(v)$, if $u, v \in C_j$ can be computed as follows:

$$P^{[1.1]} = \frac{\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-1-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}},$$

otherwise if $v \in C_j$ and $u \in C_i$ and $i \neq j$ we have:

$$P^{[1.2]} = \frac{\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}$$

Finally, for $v \in C_j$ and $u \in N(v)$ we obtain:

$$\mathbb{E}[f(u)B^{[1]}_{v,u,k,l}] = \begin{cases} 0 & \text{if } u \in C_j \\ & \text{and } \big(deg_{CS}(u) > |M|-1 \\ & \text{or } deg_j(u) > |C_j| - 1\big) \\ f(u)P^{[1.1]} & \text{if } u \in C_j \\ 0 & \text{if } u \notin C_j \\ & \text{and } \big(deg_{CS}(u) > |M|-1 \\ & \text{or } deg_j(u) > |C_j|\big) \\ f(u)P^{[1.2]} & \text{if } u \notin C_j \end{cases} \quad (4)$$

In order to compute $P^{[2]}$ we consider a complementary event $P^{[2]} = (1 - P[N(v) \cap (Q_{T^k} \cup C^l)) = \emptyset])$ and using the same combinatorial argument as for computing $P^{[1.1]}$, for $v \in C_j$ we get:

$$P^{[2]} = 1 - \frac{\binom{|M|-1-deg_{CS}(v)}{k}\binom{|C_j|-1-deg_j(v)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}},$$

and consequently we obtain:

$$\mathbb{E}[f(v)B^{[2]}_{v,k,l}] = \begin{cases} -f(v) & \text{if } \big(deg_{CS}(u) > |M|-1 \\ & \text{or } deg_j(u) > |C_j| - 1\big) \\ -f(v)P^{[2]} & \text{otherwise.} \end{cases} \quad (5)$$

The final formula combines equations (4) and (5) :

$$\begin{aligned} \mathbb{E}[\mathrm{MC}(Q_{T^k} \cup C^l, v)] &= \sum_{u \in N(v)} \Big(\mathbb{E}[f(u)B^{[1]}_{v,u,k,l}]\Big) + \mathbb{E}[f(v)B^{[2]}_{v,k,l}] \\ &= \sum_{u \in N(v) \cap C_j} f(u)\Big(\frac{\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-1-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}\Big) \\ &+ \sum_{u \in N(v) \setminus C_j} f(u)\Big(\frac{\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}\Big) \\ &- f(v)\Big(1 - \frac{\binom{|M|-1-deg_{CS}(v)}{k}\binom{|C_j|-1-deg_j(v)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}\Big). \end{aligned} \quad (6)$$

The above formula can be used to compute $\mathbb{E}_{T^k,C^l}[\mathrm{MC}(Q_{T^k} \cup C^l, v)]$ in polynomial time. Therefore, the game-theoretic network degree centrality for graph G with community structure CS can be computed in polynomial time using equation (2), which ends our proof. $\square$

5.3 Algorithm

Algorithm 1 directly implements expression (2). The expected value operator is computed using the final result of Theorem 1: equation (6). It computes the game-theoretic network degree centrality for a given graph G with community structure CS. For the sake of clarity, we assume in our algorithm that for $a < b$ we have $\binom{a}{b} = 0$, and for any a we have $\frac{a}{0} = 0$.

Algorithm 1: The coalitional semivalue

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Input: Graph $G = (V, E)$, node $v \in V$, coalition structure $CS$,
       functions $\beta$ and family of functions $\{\alpha\}$
Data: for each vertex $u \in V$ and the community $v \in C_j$:
$deg_{CS}(u)$ - the inter-community degree
$deg_j(u)$ - the intra-community degree
Output: $\phi_v$ coalitional semivalue-based degree centrality
$\phi_v \leftarrow 0$;
for $k \leftarrow 0$ to $|M| - 1$ do
    for $l \leftarrow 0$ to $|C_j| - 1$ do
        $\mathrm{MC}_{k,l} \leftarrow 0$;
        foreach $u \in N(v) \cap C_j$ do
            $\mathrm{MC}_{k,l} \leftarrow \mathrm{MC}_{k,l} + \frac{f(u)\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-1-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}$
        foreach $u \in N(v) \setminus C_j$ do
            $\mathrm{MC}_{k,l} \leftarrow \mathrm{MC}_{k,l} + \frac{f(u)\binom{|M|-1-deg_{CS}(u)}{k}\binom{|C_j|-deg_j(u)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}$;
        $\mathrm{MC}_{k,l} \leftarrow \mathrm{MC}_{k,l} - f(v)$;
        $\mathrm{MC}_{k,l} \leftarrow \mathrm{MC}_{k,l} + \frac{f(v)\binom{|M|-1-deg_{CS}(v)}{k}\binom{|C_j|-1-deg_j(v)}{l}}{\binom{|M|-1}{k}\binom{|C_j|-1}{l}}$;
        $\phi_v \leftarrow \phi_v + \beta(k)\alpha_j(l)\mathrm{MC}_{k,l}$;
```

This algorithm requires some precomputations. For each node $u \in V$ we need to calculate $deg_{CS}(v)$ and $deg_j(v)$. We can store these values using $O(|V|)$ space. Provided that it is possible to check the community of a given node in constant time, we can perform these precomputations in $O(|V| + |E|)$ time. In the worst case, the main algorithm works in $O(|V|^3)$ time.

Our next observation is that for trivial coalition structures (such as $CS = \{A\}$, or $CS = \{\{a_1\}, \{a_2\}, \ldots, \{a_n\}\}$) our algorithm computes any weighted degree-based semivalue in $O(|V|^2)$ time. Finally, we would like to note that this algorithm is easily adapted to directed networks. To this end, depending on the new definition of weighted group degree, we need to replace all instances of $deg_{CS}(u)$ and $deg_j(u)$ with their counterparts for directed networks: *in* or *out* degree.

6 SIMULATIONS

The main aim of this experiment is to compare three rankings created by three different methods: (i) one that uses weighted degree centrality and evaluates each node v by the number of neighbours it has (we denote it by $\nu_G^D(\{v\})$); (ii) one with the Shapley value-based degree centrality (denoted SV_v); and, (iii) one with the Owen value-based degree centrality (denoted OV_v), which evaluates nodes in the context of the communities they belong to and their respective power. Thus, the first two methods do not account for the existence of the community structure while the third one does.

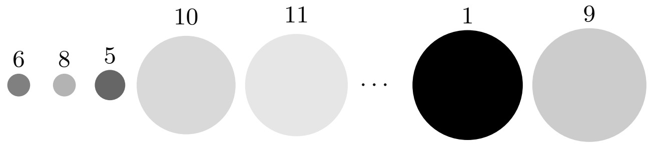

Figure 1. The relative power of communities for the first top nodes from the $\nu_G^D(\{v\})$ ranking. The power of the communities of nodes 5, 6 and 8 is significantly smaller than the power of communities of the other top nodes.

The real-life network used for simulations is a citation network that consists of $2,084,055$ publications and $2,244,018$ citation relationships.[6] This dataset is a list of publications with basic attributes (such as: title, authors, venue, or citations), and it is part of the project *ArnetMiner* being under development by Tang *et al.* [28]. All publications extracted from this dataset were categorized into 22954 unique communities representing journals, conference proceedings or single book titles using basic text mining techniques. These communities can be interpreted as scientific groups united under the same topics of interests. In our experiment we use the directed version of our algorithm and assume that $f(v) = \frac{1}{\#\text{numer of articles citing } v}$. The Shapley value-based centrality (the second method) is computed using the polynomial time algorithm introduced by Michalak *et al.* [18]. The Owen value-based centrality is computed with the modification of Algorithm 1, in which thanks to the form of the α and β (in Owen value these discrete probabilities are uniform) the complexity was reduced to $O(|V| + |E|)$.

In what follows we focus on the 11 top nodes from the basic ranking $\nu_G^D(\{v\})$. Figure 1 shows the relative power of the communities to which these nodes belong. Nodes indexed 5, 6 and 8 belong to significantly less powerful communities than nodes 1, 2, 3, 4, 7, 9, 10 and 11.

Figure 2 shows how the position of top nodes selected using $\nu_G^D(\{v\})$ changes in the SV_v and OV_v rankings. While for most nodes the perturbations are not so intensive, we observe significant downgrade of the position of nodes 5, 6 and 8 in the OV_v ranking. This demonstrates coalitional semivalues-based centrality (in this case the Owen value-based centrality) is able to recognize that these three nodes belong to much weaker communities.

Table 3. The values of different coalitional semivalues.

Solution name / Nodes's degree	17	16	11	10	9
Owen value	3.51	2.68	1.47	1.37	0.70
Owen-Banzhaf value	2.28	1.38	0.47	0.88	0.01
symmetric coalitional Banzhaf value	3.51	2.68	1.47	1.37	0.70
symmetric coalitional p-binomial semivalue ($p = \frac{1}{4}$)	4.38	3.15	1.65	2.38	1.04

The rankings of nodes may differ depending which coalitional semivalue we choose. To illustrate this fact we evaluated top 5 nodes with the highest degree centrality from Zachary Karate Club Network [29]. This network consists of 34 nodes divided into two communities. We observe in Table 3 that the ranking created with the Owen value differs with the one created with Owen-Banzhaf value at the 3rd and 4th positions.

7 RELATED WORK

Coalitional game-theory and centrality measures were first combined by Grofman and Owen [17], who introduced a centrality metric based

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[6] The database used for these experiments is available under the following link: http://arnetminer.org/citation.

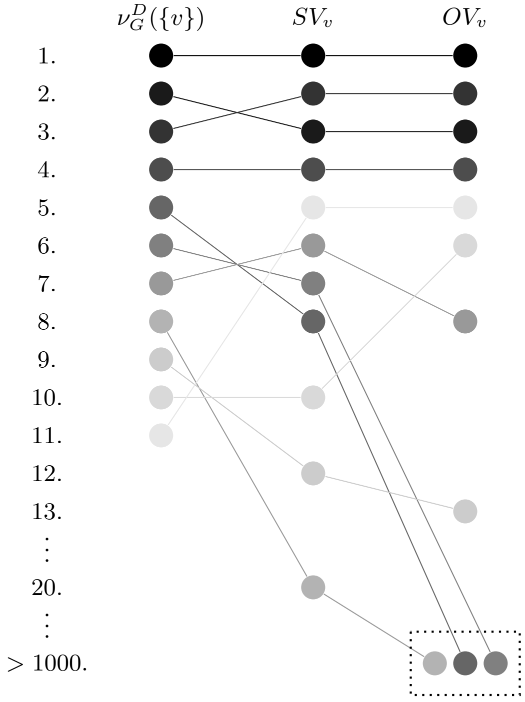

Figure 2. Three rankings of the top nodes. The OV ranking radically decreases the positions of the nodes 5, 6 and 8.

on the Banzhaf power index. The next important step in this field was made by Gomez et al. [16], who axiomatized a centrality measure based on the Shapley value and graph restricted games [20]. Semivalues as a measure of the importance of nodes were for the first time used by Amer and Gimenez [3]. Amer et al. [10] used solution concepts from generalized coalitional games [22] in order to create centralities for directed networks. Works on computational analysis of game-theoretic centrality include [18, 27, 19].

A community structure was introduced by Girvan and Newman [15]. Much literature has been devoted to defining communities [21, 1] and developing efficient algorithms for community detection within networks [15, 13]. However, the issue of how community structure influences node centrality has not yet been studied.

8 SUMMARY AND FUTURE WORK

The centrality metric proposed in this paper is the first tool that evaluates individual nodes *in the context of their communities*. This metric is based on the Owen value—a well-known concept from coalitional game theory that we generalize by introducing coalitional semivalues. Our experiments show that the rankings can significantly differ if we account for the power of the relevant communities that the nodes belong to. If the community of a node is weak, it can significantly weaken the position of the node in the ranking based on the coalitional semivalue. It also demonstrates that our polynomial time algorithm is applicable to large data sets.

In our opinion, the most interesting direction for future work is to develop the coalitional semivalue-based measures that extend other than degree centralities, especially the betweenness and the closeness centralities [14].

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REFERENCES

[1] R. Aldecoa and I. Marín, ‘Deciphering Network Community Structure by Surprise’, *PLoS ONE*, **6**(9), e24195, (2011).

[2] R. Amer, F. Carreras, and J. M. Giménez, ‘The modified banzhaf value for games with coalition structure: an axiomatic characterization.’, *Mathematical Social Sciences*, **43**(1), 45–54, (2002).

[3] R. Amer and J.M. Gimnez, ‘A connectivity game for graphs.’, *Math. Meth. of OR*, **60**(3), 453–470, (2004).

[4] H. Aziz, O. Lachish, M. Paterson, and R. Savani, ‘Power indices in spanning connectivity games’, in *Proc. of the AAIM*, pp. 55–67, (2009).

[5] Y. Bachrach and J. S. Rosenschein, ‘Power in threshold network flow games.’, *AAMAS*, **18**(1), 106–132, (2009).

[6] J. F. Banzhaf, ‘Weighted Voting Doesn’t Work: A Mathematical Analysis’, *Rutgers Law Review*, **19**, 317–343, (1965).

[7] P. Bonacich, ‘Factoring and weighting approaches to status scores and clique identification’, *J. of Mat. Soc.*, **2**(1), 113–120, (1972).

[8] F. Carreras and M. Puente, ‘Symmetric coalitional binomial semivalues’, *Group Decision and Negotiation*, **21**(5), 637–662, (2012).

[9] G. Chalkiadakis, E. Elkind, and M. Wooldridge, *Computational Aspects of Cooperative Game Theory*, Synthesis Lec. on AI and ML, 2011.

[10] M. del Pozo, C. Manuel, E. Gonzlez-Arangena, and G. Owen, ‘Centrality in directed social networks. a game theoretic approach.’, *Social Networks*, **33**(3), 191–200, (2011).

[11] P. Dubey, A. Neyman, and R. J. Weber, ‘ Value Theory Without Efficiency.’, *Mathematics of Operations Research*, **6**, 122–128, (1981).

[12] M. G. Everett and S. P. Borgatti, ‘The centrality of groups and classes’, *Journal of Mathematical Sociology*, **23**(3), 181–201, (1999).

[13] S. Fortunato, ‘Community detection in graphs’, *Physics Reports*, **486**(3-5), 75–174, (2010).

[14] L.C. Freeman, ‘Centrality in social networks: Conceptual clarification’, *Social Networks*, **1**(3), 215–239, (1979).

[15] M. Girvan and M. E. J. Newman, ‘Community structure in social and biological networks’, *in Proc. of NAS*, **99**(12), 7821–7826, (2002).

[16] D. Gómez, E. González-Arangüena, C. Manuel, G. Owen, M. Del Pozo, and J. Tejada, ‘Centrality and power in social networks: A game theoretic approach’, *Mathematical Social Sciences*, **46**(1), 27–54, (2003).

[17] B. Grofman and G. Owen, ‘A game-theoretic approach to measuring centrality in social networks’, *Social Networks*, **4**, 213–224, (1982).

[18] T. P. Michalak, K. V. Aadithya, P. L. Szczepański, B. Ravindran, and N. R. Jennings, ‘Efficient computation of the shapley value for game-theoretic network centrality.’, *J. Artif. Intell. Res.*, **46**, 607–650, (2013).

[19] T. P. Michalak, T. Rahwan, P. L. Szczepanski, O. Skibski, R. Narayanam, M. J. Wooldridge, and N. R. Jennings, ‘Computational analysis of connectivity games with applications to the investigation of terrorist networks’, *IJCAI*, (2013).

[20] R.B. Myerson, ‘Graphs and cooperation in games’, *Mathematics of Operations Research*, **2**(3), 225–229, (1977).

[21] M. E. Newman, ‘Modularity and community structure in networks’, *Proc Natl Acad Sci U S A*, **103**, 8577–8582, (2006).

[22] A. Nowak and T. Radzik, ‘The shapley value for n-person games in generalized characteristic function form’, *GEB*, **6**(1), 150–161, (1994).

[23] G. Owen, ‘Values of games with a priori unions’, in *Mathematical economics and game theory: Essays in honor of Oskar Morgenstern*, volume 141 of *LNEMS*, 76–88, Berlin, (1977).

[24] G. Owen, ‘Modification of the Banzhaf-Coleman index for games with a priori unions’, in *Power, Voting and Voting Power*, 232–238, (1982).

[25] L. S. Shapley, ‘A value for n-person games’, in *In Contributions to the Theory of Games, volume II*, 307–317, PUP, (1953).

[26] O. Skibski, T.P. Michalak, T. Rahwan, and M. Wooldridge, ‘Algorithms for the shapley and myerson values in graph-restricted games’, in *Proc. of the AAMAS*, (2014).

[27] P. L. Szczepański, T.P. Michalak, and T. Rahwan, ‘A new approach to betweenness centrality based on the shapley value’, in *Proc. of the AAMAS*, pp. 239–246, (2012).

[28] Jie Tang, Jing Zhang, Limin Yao, Juanzi Li, Li Zhang, and Zhong Su, ‘Arnetminer: Extraction and mining of academic social networks’, in *KDD’08*, pp. 990–998, (2008).

[29] W.W. Zachary, ‘An information flow model for conflict and fission in small groups’, *J. of Anthropological Research*, **33**, 452–473, (1977).