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Scoring Rules for the Allocation of Indivisible Goods[1]

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Abstract. We define a family of rules for dividing m indivisible goods among agents, parameterized by a scoring vector and a social welfare aggregation function. We assume that agents' preferences over sets of goods are additive, but that the input is ordinal: each agent simply ranks single goods. Similarly to (positional) scoring rules in voting, a scoring vector $s = (s_1, \dots, s_m)$ consists of m nonincreasing nonnegative weights, where s_i is the score of a good assigned to an agent who ranks it in position i. The global score of an allocation for an agent is the sum of the scores of the goods assigned to her. The social welfare of an allocation is the aggregation of the scores of all agents, for some aggregation function $\star$ such as, typically, $+$ or min. The rule associated with s and $\star$ maps a profile to (one of) the allocation(s) maximizing social welfare. After defining this family of rules, and focusing on some key examples, we investigate some of the social-choice-theoretic properties of this family of rules, such as various kinds of monotonicity, separability, envy-freeness, and Pareto efficiency.

1 INTRODUCTION

Fair division of a divisible good has put forth an important literature about *specific procedures*, either centralized [14] or decentralized [8]. Fair division of *a set of indivisible goods* has, perhaps surprisingly, been mainly addressed by looking for allocations that satisfy a series of properties (such as equity or envy-freeness) and less often by defining specific allocation rules. A notable exception is a series of papers that assume that each agent values each good by a positive number, the utility of an agent is the sum of the values of the goods assigned to her, and the resulting allocation maximizes social welfare; in particular, the *Santa Claus problem* [2] considers egalitarian social welfare, which maximizes the utility of the least happy agent. A problem with these rules is that they strongly rely on the assumption that the input is numerical. Now, as widely discussed in social choice, numerical inputs have the strong disadvantage that they suppose that interpersonal preferences are comparable. Moreover, from a practical designer point of view, eliciting numerical preferences is not easy: in contexts where money does not play any role, agents often feel more at ease expressing rankings than numerical utilities.

These are the main reasons why social choice – at least its subfield focusing on voting – usually assumes that preferences are expressed ordinally. Surprisingly, while voting rules defined from ordinal preferences have been addressed in hundreds of research articles, we can find only a few such papers in fair division (with the notable exception of matching, discussed below). Brams, Edelman, and Fishburn [5] assume that agents rank single goods and have additively separable preferences; they define a Borda-optimal allocation to be one that maximizes egalitarian social welfare, where the utility of an agent is the sum of the Borda scores of the objects assigned to her, and where the Borda score of object g_i for agent j ranges from 1 (when g_i is j's least preferred object) to m (when g_i is j's most preferred object). Unlike Brams et al. [5], Herreiner and Puppe [13] assume that agents should express rankings over *subsets* of goods, which, in the worst case, requires agents to express an exponentially large input, which should be avoided for obvious reasons.

One setting where it is common to use ordinal inputs is *two-sided matching*. But there, only one item is assigned to each agent, making this a rather different problem. This remark allows us to see fair division rules defined from ordinal inputs as a one-to-many extension of matching mechanisms. Examples of practical situations when one has to assign not a single, but several (sometimes many) items to each agent are common, and expressing quantitative utilities is not always feasible in such cases: composition of sport teams, divorce settlement, exploitation of Earth observation satellites (see [8] for more examples).

We start by generalizing Borda-optimal allocations [5] to arbitrary scoring vectors and aggregation functions. Beyond Borda, the scoring vectors we consider are k-approval (the first k objects get score 1 and all others get 0), lexicographicity (an item ranked in position k counts more than the sum of all objects ranked in positions $k+1$ to n), and quasi-indifference (for short, QI: all objects have roughly the same score, up to small differences). As for aggregation functions, we focus on utilitarianism ($\star = +$) and egalitarianism ($\star = \min$, as well as $\star =$ leximin, which in a *strict sense* is not an aggregation function). In Section 2, we define these allocation rules (we consider both resolute rules and irresolute rules), and focus on a few particular cases. Each of the following sections is devoted to a property or a class of properties. While the properties of voting rules have been studied extensively, this is much less the case for fair allocation of indivisible goods. Perhaps the most closely related research is [11] who study the axiomatic property of *multiwinner voting rules*, with a focus on positional scoring rules, while the relationship between multiwinner rules and resource allocation is addressed in [16].

In Section 3, we consider *separability*, which, roughly, says that if we partition the set of agents into two subsets, A_1 and A_2, where A_i collectively gets the set G_i of goods under an optimal allocation π, and if we then consider the allocation problem restricted to A_i and G_i, then the agents in A_i will get the same set G_i of goods as in π. Section 4 considers *monotonicity*: if agent i gets good g under the

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optimal allocation π, and if the rank of g is raised in i's ranking with everything else being unchanged, will i still get g? In Section 5, we look at two other forms of monotonicity, named *object monotonicity* (if some good is added, will the new allocation make all agents at least as happy as before?) and *duplication monotonicity* (which is also related to "cloning" agents). Finally, in Section 6, we consider various *consistency* and *compatibility properties*.

2 SCORING ALLOCATION RULES

Let $N = \{1,\dots,n\}$ be a set of agents and $G = \{g_1,\dots,g_m\}$ a set of indivisible goods (we will use the terms *good*, *item*, and *object* as synonyms). An *allocation* is a partition $\pi = (\pi_1,\dots,\pi_n)$, where $\pi_i \subseteq G$ is the bundle of goods assigned to agent i. We say that allocation π *gives g_i to j* if $g_i \in \pi_j$.

In the general case, to compute an optimal allocation (for some notion of optimality) we would need, for every agent, her ranking over all subsets of G. As listing all (or a significant part of) the subsets of G would be infeasible in practice, we now make a crucial assumption: *agents rank only single objects*. This assumption is not without loss of generality, and has important consequences; in particular, it will not be possible for agents to express preferential dependencies between objects. Under this assumption, a *singleton-based profile* $P = (>_1,\dots,>_n)$ is a collection of n rankings (i.e. linear orders) over G, and a *(singleton-based) allocation rule* (respectively, an *allocation correspondence*) maps any profile to an allocation (respectively, a nonempty subset of allocations). For any ranking $>$ (respectively, profile P) over G, and any subset $G' \subset G$, we will write $>_{|G'}$ (respectively, $P_{|G'}$) to denote the *restriction* of $>$ (respectively, P) to G'. Similarly, we denote the restriction of P to any subset $N' \subset N$ by $P_{|N'}$.

We now define a family of allocation rules that more or less corresponds to the family of scoring rules in voting (see, e.g., [6]).

Definition 1 *A* scoring vector *is a vector $s = (s_1,\dots,s_m)$ of real numbers such that $s_1 \geq \dots \geq s_m \geq 0$ and $s_1 > 0$. Given a preference ranking $>$ over G and $g \in G$, let $rank(g,>) \in \{1,\dots,m\}$ denote the* rank of g under $>$. *The* utility function over 2^G induced by the ranking $>$ on G and the scoring vector s *is for each bundle $X \subseteq G$ defined by $u_{>,s}(X) = \sum_{g\in X} s_{rank(g,>)}$.*

A *strictly decreasing* scoring vector s satisfies $s_i > s_{i+1}$ for each $i < m$. A scoring vector is only defined for a fixed number of objects. To deal with a variable number of objects, we introduce the notion of *extended scoring vector*, as a function mapping each integer m to a scoring vector $s(m)$ of m elements. We consider the following specific extended scoring vectors:

- Borda scoring: borda $= m \mapsto (m, m-1,\dots,1)$,[6]
- lexicographic scoring: lex $= m \mapsto (2^{m-1}, 2^{m-2},\dots,1)$,
- quasi-indifference for some extended scoring vector s: s-qi $= m \mapsto (1 + s_1(m)/M,\dots,1 + s_m(m)/M)$, with $M \gg m\cdot\max\{s_1(m),\dots,s_m(m)\} = m\cdot s_1(m)$, where M is an arbitrary and large integer.
- k-approval: k-app $= m \mapsto (1,\dots,1,0,\dots,0)$, where the first k entries are ones and all remaining entries are zero.

[6] Note that the usual definition of the Borda scoring vector in voting is $(m-1, m-2,\dots,1,0)$. Here, together with [5] we fix the score of the bottom-rank object to 1, meaning that getting it is better than nothing. For scoring voting rules, a translation of the scoring vector has obviously no impact on the winner(s); for allocation rules, however, it does. See Example 2.

In the following, we will often abuse notation and use scoring vectors and extended scoring vectors interchangeably, and omit the parameter m when the context is clear.

Note that quasi-indifference makes sense for settings where all agents should get the same number of objects (plus/minus one). An example of quasi-indifference scoring vector would be the one proposed by Bouveret and Lang [4], namely borda-qi $= (1 + m/M, 1 + (m-1)/M,\dots,1 + 1/M)$.

For example, let $G = \{a,b,c\}$ be a set of three goods and let two agents have the following preference profile: $(a >_1 b >_1 c,\ b >_2 c >_2 a)$. Let $\pi = (\{a\},\{b,c\})$. Then, for the Borda scoring vector, agent 1's bundle $\{a\}$ has value 3 and agent 2's bundle $\{b,c\}$ has value $3+2=5$.

It is important to note that *we do not claim that these numbers actually coincide, or are even close to, the agents' actual utilities* (although, in some specific domains, scoring vectors could be learned from experimental data). But this is the price to pay for defining rules from an *ordinal* input (see the Introduction for the benefits of ordinal inputs). This tradeoff is very common in voting theory: the well-studied family of *scoring rules* in voting theory (including the Borda rule) proceeds exactly the same way; voters rank alternatives, and the ranks are then mapped to scores; the winning alternatives are those that maximize the sum of scores. If we aim at maximizing actual social welfare, then we have to elicit the voters' (numerical) utilities rather than just asking them to rank objects. Caragiannis and Procaccia [9] analyze this ordinal-cardinal tradeoff in voting and show that the induced distortion is generally quite low. A reviewer pointed out that this approach also can be seen as optimizing the external perception of fairness or welfare.

The individual utilities are then aggregated using a monotonic, symmetric *aggregation function* that is to be maximized. The three we will use here are among the most obvious ones: utilitarianism (*sum*) and two versions of egalitarianism (*min* and *leximin*). Leximin refers to the (strict) lexicographic preorder over utility vectors whose components have been preordered nondecreasingly. Formally, for $x = (x_1,\dots,x_n)$, let $x' = (x'_1,\dots,x'_n)$ denote some vector that results from x by rearranging the components of x nondecreasingly, and define $x <_{\text{leximin}} y$ if and only if there is some i, $0 \leq i < n$, such that $x'_j = y'_j$ for all j, $1 \leq j \leq i$, and $x'_{i+1} < y'_{i+1}$, and $x \leq_{\text{leximin}} y$ means $x <_{\text{leximin}} y$ or $x = y$. Let leximin denote the maximum on a set of utility vectors according to $\leq_{\text{leximin}}$. For each scoring vector s, define three *allocation correspondences*:

- $F_{s,+}(P) = \text{argmax}_\pi \sum_{1\leq i\leq n} u_{>_i,s}(\pi_i)$,
- $F_{s,\min}(P) = \text{argmax}_\pi \min_{1\leq i\leq n}\{u_{>_i,s}(\pi_i)\}$, and
- $F_{s,\text{leximin}}(P) = \text{argleximin}_\pi(u_{>_1,s}(\pi_1),\dots,u_{>_n,s}(\pi_n))$,

where $P = (>_1,\dots,>_n)$ is a profile and $\pi = (\pi_1,\dots,\pi_n)$ an allocation. Whenever we write $F_{s,\star}$, we mean any one of $F_{s,+}$, $F_{s,\min}$, and $F_{s,\text{leximin}}$.

Example 2 *For $n = 3$ agents and $m = 4$ goods, $G = \{a,b,c,d\}$, let $P = (c >_1 b >_1 a >_1 d,\ c >_2 a >_2 b >_2 d,\ b >_3 d >_3 c >_3 a) = (cbad,\ cabd,\ bdca)$. Then, $F_{(4,3,2,1),\text{leximin}}(P) = \{(c,ad,b)\}$ and $F_{(3,2,1,0),\text{leximin}}(P) = \{(c,a,bd)\}$. (We omit stating "$>_i$" explicitly in the preferences, and parentheses and commas in allocations.)*

Tie-breaking: Similarly as in voting theory, an *allocation rule* is defined as the composition of an allocation correspondence and a tie-breaking mechanism, which breaks ties between allocations. One particular type of deterministic tie-breaking mechanism consists in defining it from a linear order $>_T$ over all allocations,[7] or, when N and G are not both fixed, a collection of linear orders $>_T^{N,G}$ (which we still denote by $>_T$) for all possible sets of agents and goods, N and G. We write $\pi \geq_T \pi'$ for ($\pi >_T \pi'$ or $\pi = \pi'$). As in voting, if the output of a correspondence $F(P)$ is not a singleton, then the most prioritary allocation in $F(P)$ is selected: $F^T(P) = (T \circ F)(P) = max(>_T, F(P))$.

We do not make any assumption as to how this tie-breaking relation is defined; our results hold independently of that.

One may also wonder whether it is possible to define an anonymous tie-breaking mechanism, as is common in voting. Formally, a tie-breaking mechanism $>_T$ is *anonymous* if and only if for any permutation σ over N and any pair of allocations (π, π'), we have $\pi >_T \pi' \Leftrightarrow \sigma(\pi) >_T \sigma(\pi')$, where $\sigma(\pi)$ denotes the version of π where all shares have been permuted according to σ. In fact, the answer is negative (we omit the easy proof): There is no deterministic anonymous tie-breaking mechanism.

The properties we study in the paper are primarily defined for deterministic rules. Some of them will be immediately generalizable for correspondences, and in that case we'll also discuss whether or not they hold for correspondences. However, others do not generalize in a straightforward way to correspondences.[8] For these properties, we will leave the study of whether they hold for scoring resource allocation correspondences for further research.

3 SEPARABILITY

Slightly reformulating Thomson [17], an allocation rule is *consistent* (we prefer to choose the terminology *"separable"*) if for any allocation problem and any allocation π selected by the rule, the allocation rule chooses the same allocation regardless of whether π is restricted to a subgroup of agents or when reapplying the rule to a "reduced problem" obtained by imagining the departure of any subgroup of the agents with their share. As the definition generalizes easily to allocation correspondences, we define it for both.

Definition 3 *Let $P = (>_1, \ldots, >_n)$ be a profile over a set G of goods and consider any partition of the set N of agents into two sets, N^1 and N^2, i.e., $N^1 \cup N^2 = \{1, \ldots, n\}$ and $N^1 \cap N^2 = \emptyset$. Let $\pi = (\pi_1, \ldots, \pi_n)$ and for $j \in \{1,2\}$, let $G^j = \bigcup_{i \in N^j} \pi_i$. An allocation rule F satisfies* separability *if for each P and π, $F(P_{|N^1,G^1}) = \pi^1$ and $F(P_{|N^2,G^2}) = \pi^2$, where π^i denotes the restriction of π to N^i and G^i. An allocation correspondence F satisfies* separability *if for each P and π, $\pi \in F(P)$ if and only if $\pi^1 \in F(P_{|N^1,G^1})$ and $\pi^2 \in F(P_{|N^2,G^2})$. Also, we say that a tie-breaking priority T is* separable *if $\pi^1 \geq_T \pi'^1$ and $\pi^2 \geq_T \pi'^2$ implies $\pi \geq_T \pi'$.*

Unfortunately, it looks like almost all our rules violate separability. We give a counterexample that works for many choices of $(s, \star)$.

Example 4 *Let $m = 9$, $n = 3$, $\star \in \{+, \min, \text{leximin}\}$, and s be a strictly decreasing vector. Consider the preference profile $P = (g_1g_4g_3g_6g_8g_7g_2g_5g_9,\ g_2g_5g_1g_8g_7g_3g_4g_6g_9,\ g_3g_6g_1g_2g_9g_4g_5g_7g_8)$. $F_{s,\star}(P)$ consists of the unique allocation $\pi = (g_1g_4g_8,\ g_2g_5g_7,\ g_3g_6g_9)$ for $\star \in \{\min, \text{leximin}\}$, and $F_{s,+}(P)$ consists of the unique allocation $\pi' = (g_1g_4, g_2g_5g_7g_8, g_3g_6g_9)$. The restriction of P to agents $\{1,2\}$ and goods $\{g_1, g_2, g_4, g_5, g_7, g_8\}$ is $P' = (g_1g_4g_8g_7g_2g_5,\ g_2g_5g_1g_8g_7g_4)$. For $\star \in \{\min, \text{leximin}\}$, $F_{s,\star}(P')$ consists of the unique allocation $(g_1g_4g_7,\ g_2g_5g_8) \neq (g_1g_4g_8,\ g_2g_5g_7)$, and $F_{s,+}(P')$ consists of the unique allocation $(g_1g_4g_7g_8, g_2g_5) \neq (g_1g_4, g_2g_5g_7g_8)$.*

We conjecture that (perhaps under mild conditions on s and $\star$), no positional scoring allocation rule is separable.

4 MONOTONICITY

The monotonicity properties below state that if an agent ranks a received good higher, all else being equal, then this agent does not lose this good (monotonicity) or still receives the same bundle (global monotonicity).

Definition 5 *An allocation rule F is* monotonic *if for every profile P, agent i, and good g, if $F(P)$ gives g to i, then for every profile P' resulting from P by agent i ranking g higher, leaving everything else (i.e., the relative ranks of all other objects in i's ranking and the rankings of all other agents) unchanged, it holds that $F(P')$ gives g to i. F is* globally monotonic *if for every profile P, agent i, and good g, if $F(P)$ gives g to i, then for every profile P' resulting from P by agent i ranking g higher, all else being equal, we have $F(P')_i = F(P)_i$.*

Clearly, global monotonicity implies monotonicity. These definitions extend to correspondences, but not in a unique way; therefore, we do not consider these extensions in the paper.

Theorem 6 *$F_{s,\star}^T$ is monotonic for every scoring vector s and aggregation function $\star$ (and tie-breaking priority T).*

Proof. For notational convenience, we give the proof only for $\star = +$, but it extends in a straightforward way to any aggregation function. Let $P = (>_1, \ldots, >_n)$ be a profile over a set G of goods with $g \in G$ and let $P' = (>'_1, >_2, \ldots, >_n)$ be a modified profile, where w.l.o.g. the first agent modifies her preferences such that g is ranked higher in $>'_1$ than in $>_1$, leaving everything else unchanged.

Let $F_{s,+}^T(P) = \pi = (\pi_1, \ldots, \pi_n)$ and let $g \in \pi_1$. Let $F_{s,+}^T(P') = \pi' = (\pi'_1, \ldots, \pi'_n)$. For a contradiction, suppose that $g \notin \pi'_1$. For every good $g' \neq g$, the rank of g' in $>'_1$ is either the same as or below the rank of g' in $>_1$, and since $g \notin \pi'_1$, we have $u_{>'_1,s}(\pi'_1) \leq u_{>_1,s}(\pi'_1)$. By monotonicity of utilitarian aggregation, this implies

$$u'(\pi') = u_{>'_1,s}(\pi'_1) + \sum_{i=2}^{n} u_{>_i,s}(\pi'_i) \leq \sum_{i=1}^{n} u_{>_i,s}(\pi'_i) = u(\pi'), \quad (1)$$

where u' is the social welfare with respect to the modified profile P'. Now, because $>'_1$ has been obtained by moving g upwards in $>_1$, we have $u_{>_1,s}(\pi_1) \leq u_{>'_1,s}(\pi_1)$. Again by monotonicity of utilitarian aggregation, this implies

$$u'(\pi) = u_{>'_1,s}(\pi_1) + \sum_{i=2}^{n} u_{>_i,s}(\pi_i) \geq \sum_{i=1}^{n} u_{>_i,s}(\pi_i) = u(\pi). \quad (2)$$

Since $\pi \in F_{s,+}^T(P)$ and $\pi' \in F_{s,+}^T(P')$, we have $u(\pi) \geq u(\pi')$ and $u'(\pi') \geq u'(\pi)$, which together with (1) and (2) implies $u'(\pi) = u(\pi) = u(\pi') = u'(\pi')$. Now, $u(\pi) = u(\pi')$ and $F_{s,+}^T(P) = \pi$ imply that $\pi >_T \pi'$. This, together with $u'(\pi) = u'(\pi')$, is in contradiction with $F_{s,+}^T(P') = \pi'$. ❑

[7] This choice comes with a loss of generality, as there are tie-breaking mechanisms that are not defined this way (we thank a reviewer for this remark). Also, we rule out the possibility of randomly breaking ties.

[8] This is the case for all properties expressing that an agent prefers a set of allocations to another set of allocations (and applies, e.g., to object monotonicity); for these properties there is not a unique way of generalizing the property, unlike in voting where this is well-known, e.g., for strategy-proofness.

This proof does not establish global monotonicity of $F^T_{s,\star}$; indeed, $\pi = F^T_{s,\star}(P)$ does not imply $\pi = F^T_{s,\star}(P')$ in general. We have the following result (the proof of which is omitted due to lack of space).

Proposition 7 *Let T be a separable tie-breaking priority. For each $m \geq 3$ and for each strictly decreasing scoring vector $s = (s_1,\dots,s_m)$, allocation rule $F^T_{s,+}$ is not globally monotonic.*

In order to show that $F^T_{s,\min}$ and $F^T_{s,\text{leximin}}$ do not satisfy global monotonicity, the approach of computing a winning allocation and showing that this allocation is not optimal for the modified profile seems to fail. Instead, we apply a utility-bounding approach. Let $OPT(P)$ denote the maximum egalitarian social welfare of a given preference profile P.

Theorem 8 *For each $m \geq 7$ and for each strictly decreasing scoring vector $s = (s_1,\dots,s_m)$ satisfying $s_1 - s_2 + s_3 - s_4 > s_m$, allocation rules $F^T_{s,\min}$ and $F^T_{s,\text{leximin}}$ do not satisfy global monotonicity.*

Proof. Consider the following two profiles of two agents: $P = (g_1 >_1 g_2 >_1 \cdots >_1 g_m,\ g_1 >_2 g_2 >_2 \cdots >_2 g_m)$ and $P' = (g_m >'_1 g_1 >'_1 \cdots >'_1 g_{m-1},\ g_1 >'_2 g_2 >'_2 \cdots >'_2 g_m)$. Let $\pi = F_{s,\min}(P)^T$ and, without loss of generality, let agent 1 be the agent that receives object g_m, that is, $g_m \in \pi_1$. Thus, profile P' is a valid change of P with respect to global monotonicity. We show that $F^T_{s,\min}(P') \neq \pi$. For the sake of contradiction, suppose that $\pi = F^T_{s,\min}(P')$. Then, for $i \in \{1,2\}$, we have $u_{>'_i,s}(\pi_i) \geq \max_{\widehat{\pi}} \min_{1\leq i\leq 2}\{u_{>'_i,s}(\widehat{\pi}_i)\} = OPT(P')$, where we maximize over all possible allocations $\widehat{\pi}$. Note that agent 2's preference is the same in P and P', i.e., $>_2 = >'_2$. Hence, $u_{>'_2,s}(\pi_2) = u_{>_2,s}(\pi_2)$. We distinguish between an even and an odd number of objects.

For even m: We give an allocation π^u that implies a lower bound for $OPT(P')$. Assign in P' even-numbered objects to agent 1 and odd-numbered objects to agent 2. It is clear that $u_{>'_1,s}(\pi^u_1) = u_{>'_2,s}(\pi^u_2) = \sum_{i \text{ odd}} s_i$. Thus, $u_{>'_2,s}(\pi_2) = u_{>_2,s}(\pi_2) \geq OPT(P') \geq \sum_{i \text{ odd}} s_i$. Because P is a profile of identical preferences, we have the invariant $u_{>_1,s}(\pi'_1) + u_{>_2,s}(\pi'_2) = \sum_i s_i$ for every allocation π'. This implies $u_{>_1,s}(\pi_1) \leq (\sum_i s_i) - u_{>_2,s}(\pi_2) = \sum_{i \text{ even}} s_i$. Now we give a lower-bounding allocation π^ℓ for $OPT(P)$. Assign in P the 1st (top-ranked), 5th, 9th, etc. and the 4th, 8th, etc. object to agent 1. All remaining objects go to agent 2. Then we have $u_{>_1,s}(\pi^\ell_1) = (s_1 + s_5 + s_9 + \cdots + s_{m-1-2I[4|m]} + s_4 + s_8 + s_{12} + \cdots + s_{m-2(1-I[4|m])}) > \sum_{i \text{ even}} s_i$ because $s_1 > s_2 > \cdots > s_m$, where $I[4 \mid m]$ is 1 if m is divisible by 4, and otherwise 0. For $u_{>_2,s}(\pi^\ell_2)$ the argument is analogous if $m \geq 4$. Since both agents realize more utility in π^ℓ than agent 1 in π, π is not optimal for P (contradiction).

For odd m: Our lower-bounding allocation $\tilde{\pi}^u$ for $OPT(P')$ is similar to the above except for assigning object m to agent 1. We need to consider only agent 2 because this agent realizes less utility: $u_{>'_2,s}(\tilde{\pi}^u_2) = (\sum_{i \text{ odd}} s_i) - s_m$. Assuming that π is optimal for P' as well, we have $u_{>_2,s}(\pi_2) \geq (\sum_{i \text{ odd}} s_i) - s_m$ which gives $u_{>_1,s}(\pi_1) \leq (\sum_{i \text{ even}} s_i) + s_m$. For a lower bound of $OPT(P)$, we specify $\tilde{\pi}^\ell$ as follows: Agent 1 *always* gets the 1st and 3rd object and starting with the 6th object every even-numbered object that follows. Agent 2 receives all remaining objects. Thus $u_{>_1,s}(\tilde{\pi}^\ell_1) = s_1 + s_3 + s_6 + s_8 + s_{10} + \cdots + s_{m-1} > (\sum_{i \text{ even}} s_i) + s_m$, which holds if and only if $s_1 + s_3 > s_2 + s_4 + s_m$. For agent 2, we have $u_{>_2,s}(\tilde{\pi}^\ell_2) = s_2 + s_4 + s_7 + \cdots + s_m > \sum_{i \text{ even}} s_i$ because of $s_1 > s_2 > \cdots > s_m$, if $m \geq 7$. It follows that π cannot be optimal for P (contradiction).

These results hold for $F^T_{s,\text{leximin}}$ as well because we take, without loss of generality, $\pi = F^T_{s,\text{leximin}}(P)$ with $g_m \in \pi_1$ and show that it is not even optimal under egalitarian social welfare in P', and hence cannot be optimal under leximin. ❑

Corollary 9 *For each scoring vector $s \in \{\text{borda}, \text{lex}\}$ for $m \geq 7$ goods, allocation rules $F^T_{s,\min}$ and $F^T_{s,\text{leximin}}$ do not satisfy global monotonicity. In addition, for each extended scoring vector s satisfying $s_1(m) > s_2(m) > \cdots > s_m(m)$ for even $m \geq 4$, allocation rules $F^T_{\text{s-qi},\min}$ and $F^T_{\text{s-qi},\text{leximin}}$ do not satisfy global monotonicity either.*

5 OBJECT AND DUPLICATION MONOTONICITY AND CLONING

Object monotonicity is a dynamic property where additional goods are to be distributed. This means that when new objects are added, no agent is worse off afterwards. In order to define this notion, since some properties need comparability of bundles of goods, we lift agent i's linear order $>_i$ to a strict partial order $\succ_i$ over 2^G by requiring monotonicity ($A \supset B \implies A \succ_i B$) and pairwise dominance (for all $A \subseteq G \setminus \{x,y\}$, $A \cup \{x\} \succ_i A \cup \{y\}$ if $x >_i y$). For strict partial orders we then follow the approach taken by Brams and King [7], Brams, Edelman, and Fishburn [5], and Bouveret, Endriss, and Lang [3]: We distinguish between properties holding *possibly* (i.e., for *some* completion of the partial preferences) and *necessarily* (i.e., for *all* completions).

Definition 10 *Let $\succ$ be a strict partial order over 2^G. We say A is* possibly preferred *to B, $A \succ^{\text{pos}} B$, if there exists a linear order $\succ^*$ refining $\succ$ such that $A \succ^* B$. Analogously, A is* necessarily preferred *to B, $A \succ^{\text{nec}} B$, if for all linear orders $\succ^*$ refining $\succ$ we have $A \succ^* B$. Allowing indifference, we extend $\succ^{\text{pos}}$ to $\succeq^{\text{pos}}$ and $\succ^{\text{nec}}$ to $\succeq^{\text{nec}}$.*

Now, we are ready to define possible and necessary object monotonicity. *These properties are defined for deterministic rules only.*

Definition 11 *Let $P = (>_1,\dots,>_n)$ be a profile over the set G of goods and let $P' = (>'_1,\dots,>'_n)$ be a profile that is obtained by adding one more good g to the set of goods, and such that the restriction of P' to G is P. An allocation rule F satisfies* possible (respectively, necessary) object monotonicity *if for all P over G, P' such that P is the restriction of P' over G, and all i, we have $F(P')_i \succeq^{\text{pos}}_i F(P)_i$ (respectively, $F(P')_i \succeq^{\text{nec}}_i F(P)_i$).*

Proposition 12 *For all tie-breaking priorities T, $F^T_{s,+}$ satisfies possible object monotonicity for all scoring vectors s for $n = 2$ agents, yet does not do so for all $n \geq 3$ and strictly decreasing scoring vectors s.*

Proof. We first give a counterexample for $n = 3$; it extends easily to more agents. Let $m = 5$, $G = \{a,b,c,d,e,f,g\}$ and $P = (abcde,\ bcdea,\ abcde)$. As $F_{s,+}(P) = \{(a,\ bcde,\ \emptyset), (\emptyset,\ bcde,\ a)\}$, w.l.o.g., let $F^T_{s,+}(P) = (a,\ bcde,\ \emptyset)$. Now, add two more goods, f and g, to G and let $P' = (abcdefg,\ bcdfgea,\ fgabcde)$. We have $F^T_{s,+}(P') = (ae,\ bcd,\ fg)$: we see that $F^T_{s,+}(P')_2 \not\succeq^{\text{pos}}_2 F^T_{s,+}(P)_2$.

For $n = 2$, let P be a two-agent profile, and P' a new one obtained by adding one more good g. Let $rank^Q_{>_i}(g_j)$ denote the rank of g_j under $>_i$ with respect to profile Q. Assume that $F^T_{s,+}(P) = (\pi_1,\pi_2)$ and $F^T_{s,+}(P') = (\pi'_1,\pi'_2)$. W.l.o.g., we can assume that $rank^{P'}_{>_1}(g) > rank^{P'}_{>_2}(g) = k$. It is easy to see that g and every object in π_2 that has rank less than k will be added to π'_2. This implies that it does not matter if agent 2 gets more goods of ranking greater than k or not, $\pi'_2 \succeq^{\text{pos}}_2 \pi_2$. We now prove that $\pi'_1 \supseteq \pi_1$. First, it is obvious that π'_1 contains objects belonging to π_1 that are ranked above g. Thus, it suffices to prove that, for every $g^* \in \pi_1$ such that $rank^{P'}_{>_1}(g^*) > rank^{P'}_{>_1}(g)$, it holds that $rank^{P'}_{>_1}(g^*) \leq rank^{P'}_{>_2}(g^*)$, and thus $g^* \in \pi'_1$. Suppose that $rank^{P'}_{>_2}(g^*) < rank^{P'}_{>_2}(g)$. It follows $rank^{P}_{>_2}(g^*) = rank^{P'}_{>_2}(g^*) < rank^{P'}_{>_2}(g) < rank^{P'}_{>_1}(g) < rank^{P'}_{>_1}(g^*)$. As $rank^{P}_{>_1}(g^*)+1 = rank^{P'}_{>_1}(g^*)$, it follows $rank^{P}_{>_1}(g^*) > rank^{P}_{>_2}(g^*)$ and this is a contradiction with the fact that $g^* \in \pi_1$. Therefore, $rank^{P'}_{>_2}(g^*) > rank^{P'}_{>_2}(g)$ and this implies $rank^{P'}_{>_2}(g^*) = rank^{P}_{>_2}(g^*) + 1 \geq rank^{P}_{>_1}(g^*) + 1 = rank^{P'}_{>_1}(g^*)$. This completes the proof. ❑

Necessary object monotonicity might not be true even with only two agents for $F^T_{+,s}$ for some tie-breaking mechanism T. This can be shown by a counterexample (omitted due to lack of space).

Monotonicity in agents has a natural translation in terms of voting power: to give more voting power to a voter, one can just allow her to vote twice (or more). In other words: duplicating a voter will give more weight to her ballot, and give her a higher chance to be heard. This property has a natural translation to the resource allocation context: informally, two agents having the same preferences will get a better share together than if they were only one participating in the allocation process. More formally:

Definition 13 *Let $P = (>_1, \ldots, >_n)$ be a profile over G and $P' = (>_1, \ldots, >_n, >_{n+1})$ be its extension to $n+1$ agents, where $>_{n+1} = >_n$. An allocation rule F satisfies* possible duplication monotonicity *if $F(P')_n \cup F(P')_{n+1} \succeq^{\text{pos}}_i F(P)_n$; and it satisfies* necessary duplication monotonicity *if $F(P')_n \cup F(P')_{n+1} \succeq^{\text{nec}}_i F(P)_n$.*

It turns out that several scoring allocation rules satisfy at least possible duplication, provided that we use "duplication-compatible" tie-breaking rules, namely, rules T that satisfy the following property: let π and π' be two allocations on $(>_1, \ldots, >_n, >_{n+1})$ (n and $n+1$ being a duplicated agent as above); then $\pi >^{n+1}_T \pi' \Rightarrow (\pi_1, \ldots, \pi_n \cup \pi_{n+1}) >^n_T (\pi'_1, \ldots, \pi'_n \cup \pi'_{n+1})$. For such tie-breaking rules we have:

Theorem 14 *For each scoring vector s, $F_{s,+}$ satisfies possible and necessary duplication monotonicity, and $F_{\text{s-qi,leximin}}$ and $F_{\text{lex,leximin}}$ both satisfy possible duplication monotonicity.*

Proof. For $F_{s,+}$, each object goes to who ranks it best. Every object that goes to agent n in the first profile will go to either n or $n+1$ in the second one (this is also guaranteed by the duplication-compatible tie-breking rule in case of ties with other agents). n and its two duplicated versions will thus get exactly the same objects, hence the result.

For $F_{\text{s-qi,leximin}}$, each agent will get at least between $\lfloor m/n \rfloor$ and $\lfloor m/n \rfloor + 1$ objects. Since $2 \cdot \lfloor m/(n+1) \rfloor > \lfloor m/n \rfloor + 1$, the two duplicated agents n and $n+1$ will receive strictly more objects than original agent n, hence proving the possible duplication monotonicity.

For $F_{\text{lex,leximin}}$, every optimal allocation is such that the minimum among every agent i of the rank $f(i)$ of the first object received by i is maximal. Moreover, if i is not among the agents whose $f(i)$ is minimal, then agent i only receives one object (and is satisfied with it). If original agent n is in this case, the duplicated agents n and $n+1$ will together be possibly better off with their new share, since they will either receive at least one object each if there are enough objects, or only one of them will receive the same object as original agent n if this is not the case. If original agent n is among the agents whose $f(i)$ is minimal, then either it is possible to give a better object than $f(i)$ to one of the duplicate agents n and $n+1$ (in this case we are done), or it is not possible, in which case one of the duplicate agents will receive $f(i)$, and the other, whose first object cannot be as high as $f(i)$, will receive all the remaining objects (including the former ones of agent n), so both of them will be at least as satisfied as before. ❑

False-name manipulation has been studied in voting [10, 19], cooperative game theory [1, 15], pseudonymous bidding in combinatorial auctions [20], and, somewhat relatedly, cloning has been studied in voting [18, 12]. Applying this setting to resource allocation, we now assume that agents can participate with multiple identities at the same time. Each of an agent's clones will have the same preferences as this agent. As they are *from the point of view of the agents*, we assume that each agent knows its own linear order over 2^G.

Definition 15 *Let $P = (>_1, \ldots, >_n)$ be a profile of linear orders over G and $\succ_i$ agent i's linear order over 2^G extending $>_i$. An allocation rule F is* susceptible to cloning of agents at *P* by agent *i* with $\succ_i$ *if there exists a nonempty set C_i of clones of i (each with the same linear order $>_i$) such that $\bigcup_{j \in C_i \cup \{i\}} \pi'_j \succ_i \pi_i$, where $\pi = (\pi_1, \ldots, \pi_n) = F(P)$, P' is the extension of P to the clones in C_i, and $\pi' = (\pi'_1, \ldots, \pi'_{n+\|C_i\|}) = F(P')$.*

Proposition 16 *If $m \geq 4$ and $m > n$, then for each strictly decreasing scoring vector $s = (s_1, \ldots, s_m)$, allocation rules $F^T_{s,\min}$ and $F^T_{s,\text{leximin}}$ are susceptible to cloning.*

We omit the proof due to lack of space.

6 CONSISTENCY AND COMPATIBILITY

Our scoring allocation rules are based on the maximization of a collective utility defined as the aggregation of individual utilities. An orthogonal classical approach is to find an allocation that satisfies a given (Boolean) criterion. Among the classical criteria, *envy-freeness* states that no agent would be better off with the share of another agent than it is with its own share, and a *Pareto-efficient* allocation cannot be strictly improved for at least one agent without making another agent worse-off. A natural question is to determine to which extent the scoring allocation rules are compatible with these criteria. More formally:

Definition 17 *Let P be a profile and let X be a property on allocations. An allocation correspondence F is X*-consistent *(respectively, X*-compatible*) if it holds that if there exists an allocation satisfying X for P, then all allocations in $F(P)$ satisfy X (respectively, there is an allocation in $F(P)$ that satisfies X).*

The interpretation is as follows: if F is X-consistent, then no matter which tie-breaking rule is used, an allocation satisfying X will always be found by the allocation rule if such an allocation exists. If F is X-compatible, it means that a tie-breaking rule which is consistent with X (that is: if $\pi \models X$ and $\pi' \not\models X$ then $\pi >_T \pi'$) is needed to find for sure an allocation satisfying X when there is one. Obviously, any X-consistent rule is also X-compatible.

We will now investigate the compatibility and consistency of the scoring rules for Pareto efficiency and envy-freeness. However, these two criteria, which are initially defined for *complete* preorders on 2^G, need to be adapted to deal with *incomplete* preferences.[9] For that, we borrow the following adaptation from [3]. First, given a linear order $\succ$ on G, we say that a mapping $w : G \to R^+$ is *compatible* with $\succ$ if for all $g, g' \in G$, we have $g \succ g'$ if and only of $w(g) > w(g')$; next, given

[9] Recall that we only know the preferences on *singletons* of objects, which have to be lifted to 2^G for the raw criteria to be directly applicable.

$A, B \subseteq G$, we say that $A \succeq^{pos} B$ if $\sum_{g \in A} w(g) \geq \sum_{g \in B} w(g)$ for some w compatible with $\succ$, and that $A \succeq^{nec} B$ if $\sum_{g \in A} w(g) \geq \sum_{g \in B} w(g)$ for all w compatible with $\succ$. Then:

Definition 18 *Let $(\succ_1, \ldots, \succ_n)$ be a profile of strict partial orders over 2^G and let π, π' be two allocations. We say (1) π'* possibly Pareto-dominates *π if $\pi'_i \succeq_i^{\text{pos}} \pi_i$ for all i and $\pi'_j \succ_j^{\text{pos}} \pi_j$ for some j; (2) π'* necessarily Pareto-dominates *π if for all $\pi'_i \succeq_i^{\text{nec}} \pi_i$ for all i and $\pi'_j \succ_j^{\text{nec}} \pi_j$ for some j; (3) π is* possibly Pareto-efficient (PPE) *if there is no allocation π' that necessarily Pareto-dominates π; (4) π is* necessarily Pareto-efficient (NPE) *if there is no allocation π' that possibly Pareto-dominates π; (5) π is* possibly envy-free (PEF) *if for every i and j, $\pi_i \succeq_i^{\text{pos}} \pi_j$; (6) π is* necessarily envy-free (NEF) *if for every i and j, $\pi_i \succeq_i^{\text{nec}} \pi_j$.*[10]

An important question is, given a profile P, whether or not there exist a scoring vector s and an aggregation function $\star$ such that the allocation correspondence $F_{s,\star}$ is X-consistent or X-compatible, where $X \in \{\text{NEF, NPE}\}$. While this question is not answered yet in general, we can first observe that $F_{s,+}$ is not NEF-consistent for strictly decreasing scoring vectors. We can also prove that these properties cannot be guaranteed for some of the specific scoring vectors considered here with min or leximin aggregation. Note that if $F_{s,\star}$ is not X-compatible then it is not X-consistent, but the converse is not always true.

Proposition 19 *Let $\star \in \{\min, \text{leximin}\}$. (1) $F_{\text{lex},\star}$ is neither NEF-compatible nor NPE-compatible. (2) $F_{s,\star}$ is neither NEF-consistent nor NPE-compatible for $s \in \{\text{borda}, \text{borda-qi}\}$. (3) $F_{k\text{-app},\star}$ is neither NEF-consistent nor NPE-consistent.*

Proof. We prove the claim for the case $\star = \min$ only, since the case $\star = $ leximin is similar.

(1) Let $P = (g_1g_2g_3g_4g_6g_5,\ g_5g_2g_4g_3g_1g_6)$. An NEF allocation is $(g_1g_3g_6,\ g_2g_4g_5)$. $F_{\text{lex},\min}(P)$ outputs $(g_1g_3g_4g_6,\ g_2g_5)$ as the unique optimal allocation, which is neither NEF nor NPE.

(2) Let $P = (g_1g_2g_3g_4g_5g_6g_7g_8,\ g_5g_1g_6g_2g_7g_3g_8g_4)$. There is an NEF allocation in which agent 1 receives the four most preferred objects and agent 2 receives the remaining ones. Obviously, there is also an NPE allocation (e.g., giving everything to agent 1). $F_{\text{borda},\min}(P)$ outputs $\pi^1 = (g_2g_3g_4g_6,\ g_1g_5g_7g_8)$, $\pi^2 = (g_1g_2g_4g_8,\ g_3g_5g_6g_7)$, $\pi^3 = (g_1g_2g_3,\ g_4g_5g_6g_7g_8)$, $\pi^4 = (g_2g_3g_4g_7g_8,\ g_1g_5g_6)$, and $\pi^5 = (g_1g_3g_4g_7,\ g_2g_5g_6g_8)$, whose social welfare is 21 each, but only the last one is NEF. Furthermore, one can easily check that none of these allocations is NPE. Similar arguments work for $F_{\text{borda-qi},\min}(P)$.

(3) It is easy to see that allocation π^1 above is among the optimal ones for $F_{7\text{-app},\min}(P)$, and hence that $F_{k\text{-app},\min}(P)$ is neither NEF-consistent nor NPE-consistent. ❑

Proposition 20 *If $n = m$, for each scoring vector s, $F_{s,\min}$ and $F_{s,\text{leximin}}$ are NEF-compatible (and even NEF-consistent for strictly decreasing s) and NPE-compatible.*

Proof. If $n = m$, then the only NEF allocations are such that all the agents receive their most preferred item. This allocation is obviously also among the optimal ones (or exactly the optimal one for strictly decreasing s). Moreover, there is at least one allocation π giving one object to each agent among the $F_{s,\text{leximin}}$ (and hence $F_{s,\min}$) optimal ones. Either π is NPE, or there is an NPE allocation π' possibly Pareto-dominating π (hence also giving one object to each agent). π' is obviously also among the $F_{s,\text{leximin}}$ optimal allocations, hence proving that $F_{s,\text{leximin}}$ and $F_{s,\min}$ are NPE-compatible. ❑

[10] For $i \neq j$, $\pi_i \succeq_i^{\text{pos}} \pi_j$ and $\pi_i \succ_i^{\text{pos}} \pi_j$ ($\pi_i \succeq_i^{\text{nec}} \pi_j$ and $\pi_i \succ_i^{\text{nec}} \pi_j$) are equivalent, as the bundles to be compared are always disjoint.

7 CONCLUDING REMARKS

Generalizing earlier work [7, 5], we have defined a family of rules for the allocation of indivisible goods to agents that are parameterized by a scoring vector and an aggregation function. We have discussed a few key properties, and for each of them we have given some positive as well as some negative results about their satisfaction by scoring allocation rules. The relatively high number of negative results should be balanced against the satisfaction of several important properties (including monotonicity) together with the simplicity of these rules. And anyway, defining allocation rules of indivisible goods from ordinal inputs on other principles does not look easy at all. Our results are far from being complete: for many properties we do not have an exact characterization of the scoring allocation rules that satisfy them, and obtaining such exact characterizations is left for further research.

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