ECAI 2014
T. Schaub et al. (Eds.)

doi:10.3233/978-1-61499-419-0-627

From analogical proportions in lattices to proportional analogies in formal concepts

Laurent Miclet and **Nelly Barbot**[1] and **Henri Prade**[2]

Abstract. The paper provides an attempt at bridging formal concept analysis and the modeling of analogical proportions (i.e., statements of the form "a is to b as c is to d"). A suitable definition for analogical proportions in non distributive lattices is proposed and then applied to concept lattices. This enables us to compute what we call proportional analogies that establish analogies on a proportional basis between pairs (a, b) and (c, d) when a and c belong to a domain and b and d to another domain (as in "Moby Dick is to Herman Melville as Alice in Wonderland is to Lewis Carroll").

1 Introduction

Aristotle [1] discussed metaphors as particular forms of analogies. Let us quote an excerpt that includes often cited examples:

Thus the cup is to Dionysus as the shield to Ares. The cup may, therefore, be called 'the shield of Dionysus,' and the shield 'the cup of Ares.' Or, again, as old age is to life, so is evening to day. Evening may therefore be called 'the old age of the day,' and old age, 'the evening of life,'or, in the phrase of Empedocles,'life's setting sun.'

The sentence "the cup is to Dionysus as the shield to Ares" is an example of analogies between the two pairs (*cup, Dionysus*) and (*shield, Ares*), where each pair is made of two elements belonging to two different categories, namely objects and (Greek) gods respectively. This example may suggest a linkage between two particular areas of research in artificial intelligence, namely, on the one hand, formal concept analysis [6] where formal concepts are extracted from a formal context which is nothing but a binary relation linking two distinct universes, and on the other hand, analogical reasoning, in particular based on analogical proportions [7, 15]. It is worth noting that these two areas are respectively related to two basic mental activities: categorization and analogy making.

An analogical proportion is a statement of the form "x is to y as z is to t", which may be denoted $x : y :: z : t$. The formal modeling of analogical proportions has raised the interest of some researchers for more than a decade now, who have proposed algebraic [9, 17] or logical [11] approaches. In the latter view, the analogical proportion is understood as "x differs from y as z differs from t", and "y differs from x as t differs from z". This is closely related to the idea that the pair (x, y) is analogous to the pair (z, t) [8], both pairs playing a symmetrical role. Analogical proportions have been proved to be a fruitful basis for an original approach to classification in machine learning [10] where the analogical proportions that are looked for stand between vectors of attributes values describing objects. The analogical proportions that are considered are then between four (Boolean) values pertaining to *the same* attribute and referring to four objects.

As already illustrated by the Aristotle's example, x, y, z, t in an analogical proportion may not belong to the same domain; namely, x and z may belong to a domain clearly distinct from the one to which y and t belong. We shall simply call this latter type of proportion *proportional analogy*, or more shortly *analogy*. While it is natural that analogical proportions on a unique domain satisfies central permutation ($x : y :: z : t$ is equivalent to $x : z :: y : t$) as a numerical proportion does, central permutation seems debatable for proportional analogies. Indeed, while "the cup is to Dionysus as the shield to Ares" sounds right, "the cup is to the shield as Dionysus to Ares" seems a bit strange.

Such proportional analogies can be looked for in a formal context, that is the Cartesian product of two sets, usually understood as a set of objects and a set of attributes. Then a formal concept is a pair $(\mathbf{o}, \mathbf{a})$, where $\mathbf{o}$ is a subset of objects and $\mathbf{a}$ is a subset of properties. The set of formal concepts associated with a formal context can be organized in a lattice structure [6]. This is a non-distributive lattice in general. Interestingly enough, Mary Hesse in an early, visionary paper [8] foresaw the interest of looking for analogical proportions in non-distributive lattices, at a time where formal concept analysis was not existing and where the formal modeling of analogical proportions was still in its infancy.

Analogical proportions have been defined not only on Boolean lattices, but also in general lattices [17, 2], and studied in distributive lattices [2]. The present paper investigates the notion of analogical proportions in non-distributive lattices (where some equivalences between different views no longer hold), and then studies analogical proportions in concept lattices, before discussing how to produce proportional analogies and being able to answer questions like "what is the cup of Ares?" by "the shield". The ambition of this paper is thus to build a formal bridge between analogical proportions and formal concepts. Some preliminary work connecting the two areas can be found in [12], but where analogical proportions were not considered between concepts, but only between objects.

The paper is organized as follows. Section 2 provides the basics about the formal modeling of analogical proportions in lattices. Section 3 introduces weak analogical proportions (WAP) that still make sense in non-distributive lattices. Section 4 studies WAP in concept lattices, and then describes the process of building proportional analogies, before concluding in Section 5.

2 Background

Since the paper deals with analogical proportions in concept lattices, we first introduce analogical proportions before providing a short refresher on lattice structures.

[1] IRISA, University of Rennes 1, France. Email: {Laurent.Miclet, Nelly.Barbot}@univ-rennes1.fr
[2] IRIT, University of Toulouse 3, France & QCIS, University of Technology, Sydney, Australia. Email: prade@irit.fr

2.1 Analogical proportions

2.1.1 Axiomatic definition

Analogical proportions are characterized by three axioms. The first two axioms acknowledge the symmetrical role played by the pairs (x, y) and (z, t) in the proportion 'x is to y as z is to t', and enforce the idea that y and z can be interchanged if the proportion is valid, just as in the equality of two numerical ratios where means can be exchanged. A third (optional) axiom, called determinism, insists on the unicity of the solution $t = y$ to the analogical proportion equation in t: $x : y :: x : t$. These axioms are studied in [9].

Definition 1 (Analogical proportion) *An analogical proportion*[3] *(AP) on a set X is a quaternary relation on X, i.e. a subset of X^4. An element of this subset, written $x : y :: z : t$, which reads 'x is to y as z is to t', must obey the following two axioms:*

1) Symmetry of 'as'*:* $x : y :: z : t \Leftrightarrow z : t :: x : y$
2) Exchange of means*:* $x : y :: z : t \Leftrightarrow x : z :: y : t$

Then, thanks to symmetry and central permutation, it can be easily seen that $x : y :: z : t \Leftrightarrow t : y :: z : x$ should also hold (exchange of the extremes). From the first two axioms, seven other formulations are equivalent to the canonical form $x : y :: z : t$, namely $z : t :: x : y$, , $y : x :: t : z$, $t : z :: y : x$, $z : x :: t : y$, , $t : y :: z : x$, $x : z :: y : t$ and $y : t :: x : z$.

2.1.2 Definition by factorization

Stroppa and Yvon [16, 18] have given another definition of the analogical proportion, based on the notion of *factorization*, when the set of objects is a commutative semigroup $(X, \oplus)$.

Definition 2 *$(x, y, z, t) \in X^4$ is an AP $(x : y :: z : t)$ if:*

1) either $(y, z) \in \{(x, t), (t, x)\}$,
2) or there exists $(x_1, x_2, t_1, t_2) \in X^4$ such that $x = x_1 \oplus x_2$, $y = x_1 \oplus t_2$, $z = t_1 \oplus x_2$ and $t = t_1 \oplus t_2$.

This definition satisfies the two basic axioms of the analogical proportion (Definition 1). For example, in $(X, \oplus) = (\mathbb{N}^+, \times)$, with $x_1 = 2$, $x_2 = 3$, $t_1 = 5$ and $t_2 = 7$, one has $(2 \times 3) : (2 \times 7) :: (5 \times 3) :: (5 \times 7)$, i.e. $6 : 14 :: 15 : 35$, a numerical geometric analogical proportion. Note that this particular proportion corresponds equivalently to the equality: $6 \times 35 = 14 \times 15$.

2.2 Lattices

Lattices are mathematical structures commonly encountered in the semantics of representation and programming languages, in formal concept analysis, machine learning, data mining, and in other areas of computer sciences.

Definition 3 *$(L, \vee, \wedge, \leq)$ is a* lattice *when [5]:*

i) L has at least two elements,
ii) $\wedge$ and $\vee$ are two binary internal operations, both idempotent, commutative, associative, and satisfying the absorption laws: $u \vee (u \wedge v) = u \wedge (u \vee v) = u$ for all u and v in L.

[3] When there is no ambiguity, an analogical proportion is also called a *proportion*.

Equivalently $(L, \vee, \wedge, \leq)$ can be defined as a poset in which every couple of elements has a supremum and an infimum according to $\leq$. We have in particular $x \wedge y \leq x \leq x \vee y$ for every $(x, y) \in L^2$.

A lattice is *distributive* when $u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w)$, or equivalently $u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w)$ for every $(u, v, w) \in L^3$. A *bounded* lattice has a greatest (or maximum) and least (or minimum) element, denoted $\top$ and $\bot$. It is *complemented* if each element x has a complementary y such that $x \wedge y = \bot$ and $x \vee y = \top$. A distributive, bounded and complemented lattice is called a *Boolean* lattice. A lattice is *complete* when all its subsets have a supremum and an infimum.

Examples. (a) $(2^\Sigma, \cap, \cup, \subseteq)$, where Σ is a finite set, is a Boolean lattice. (b) $(\mathbb{N}^+, \gcd, \mathrm{lcm}, |)$ where $(x \mid y)$ iff x divides y is a distributive lattice, with the minimum element 1 but no maximum element. (c) The set $\mathcal{S}$ of closed intervals on $\mathbb{R}$, including $\varnothing$ and $\mathbb{R}$, is a non-distributive lattice when $\wedge$ is the intersection and $[a, b] \vee [c, d] = [\min(a, c), \max(b, d)]$.

3 Analogical proportions in lattices: Basics

In this section, we are interested in studying how the definition of an analogical proportion by factorization applies to lattices. In particular we are wondering whether the equivalence of the two formulations in the preceding examples can be transposed to this algebraic structure.

3.1 Definition

As in a lattice $(L, \vee, \wedge, \leq)$, both $(L, \vee)$ and $(L, \wedge)$ are commutative semigroups, we can derive the following result from Definition 2.

Proposition 1 *$(x, y, z, t) \in L^4$ is an AP $(x : y :: z : t)$ iff:*

1) there exists $(x_1, x_2, t_1, t_2) \in L^4$ such that $x = x_1 \vee x_2$, $y = x_1 \vee t_2$, $z = t_1 \vee x_2$ and $t = t_1 \vee t_2$,
2) and there exists $(x'_1, x'_2, t'_1, t'_2) \in L^4$ such that $x = x'_1 \wedge x'_2$, $y = x'_1 \wedge t'_2$, $z = t'_1 \wedge x'_2$ and $t = t'_1 \wedge t'_2$.

Note that if $x_2 = t_2$ then $y = x$ and $z = t$ and if $x_1 = t_1$ then $y = t$ and $z = x$. Hence we can have $(y, z) = (x, t)$ or $(y, z) = (t, x)$.

Examples. (a) In $(\mathbb{N}^+, \gcd, \mathrm{lcm}, |)$, we have $(20 : 4 :: 60 : 12)$, with $x_1 = 20$, $x_2 = t_1 = 60$, $t_2 = 12$, $x'_1 = t'_2 = 4$, $x'_2 = 20$ and $t'_1 = 12$. (b) In the lattice $\mathcal{S}$ of closed intervals on $\mathbb{R}$, we have $([0, 3] : \{3\} :: [0, 4] : [3, 4])$ with $x_1 = \{3\}$, $x_2 = \{0\}$, $t_1 = [3, 4]$, $t_2 = \varnothing$, $x'_1 = [0, 3]$, $x'_2 = [0, 4]$, $t'_1 = [0, 4]$ and $t'_2 = [3, 4]$.

Proposition 2 *(x, y, z, t) in L^4 is an AP $(x : y :: z : t)$ iff:*

$$\begin{array}{lll} x = (x \wedge y) \vee (x \wedge z) & & x = (x \vee y) \wedge (x \vee z) \\ y = (x \wedge y) \vee (y \wedge t) & & y = (x \vee y) \wedge (y \vee t) \\ z = (z \wedge t) \vee (x \wedge z) & & z = (z \vee t) \wedge (x \vee z) \\ t = (z \wedge t) \vee (y \wedge t) & & t = (z \vee t) \wedge (y \vee t) \end{array}$$

Proof. $(\Rightarrow)$. Let us show that $x = (x \wedge y) \vee (x \wedge z)$. Since $x = x_1 \vee x_2$ and $y = x_1 \vee t_2$, we have $x_1 \leq x$ and $x_1 \leq y$. Then $x_1 \leq x \wedge y$. Similarly, factor x_2 satisfies $x_2 \leq x \wedge z$. Hence, $x \leq (x \wedge y) \vee (x \wedge z)$. Besides, x being greater than $(x \wedge y)$ and $(x \wedge z)$, $(x \wedge y) \vee (x \wedge z) \leq x$. The antisymmetry of $\leq$ implies that $x = (x \wedge y) \vee (x \wedge z)$. We show the other equalities in the same manner.
$(\Leftarrow)$. Taking $x_1 = x \wedge y$, $x_2 = x \wedge z$, $t_1 = z \wedge t$ and $t_2 = y \wedge t$ show directly that there exist factors satisfying Proposition 1. □

Proposition 3 *Let $(x, y, z, t) \in L^4$, if $(x : y :: z : t)$ then*

$$x \vee t = y \vee z \text{ and } x \wedge t = y \wedge z. \tag{1}$$

If $(L, \vee, \wedge, \leq)$ is distributive, the converse is true.

Proof. Equation (1) can be easily checked using the factorisations given by Proposition 2. Conversely, by absorption law and distributivity, we have $x = x \wedge (x \vee t) = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The other equations of Proposition 2 can be obtained in a similar way. □

Comment. Proposition 3 is not an equivalence in general. For example, in the lattice $\mathcal{S}$, $x = [2, 3]$, $y = [2, 6]$, $z = [8, 9]$ and $t = [6, 9]$ satisfy $x \vee t = y \vee z$ and $x \wedge t = y \wedge z$ but $x \neq (x \vee y) \wedge (x \vee z)$.

3.2 Determinism

The first and second axioms of Definition 1 are straightforwardly verified by Proposition 1. What about the third axiom?

Proposition 4 (determinism) *Let x and y be two elements of a lattice, the equation in z: $(x : x :: y : z)$ has the unique solution $z = y$. This is also true for the equation $(x : y :: x : z)$.*

Proof. Let z be such that $(x : x :: y : z)$, we have

$$y = (y \wedge z) \vee (x \wedge y) \quad (2)$$
$$y = (y \vee z) \wedge (x \vee y) \quad (3)$$
$$z = (y \wedge z) \vee (x \wedge z) \quad (4)$$
$$z = (y \vee z) \wedge (x \vee z) \quad (5)$$

from Proposition 2. Consequently, using (3) and absorption law,

$$x \wedge y = x \wedge (y \vee z) \wedge (x \vee y) = x \wedge (y \vee z)$$

and similarly, using (5)

$$x \wedge z = x \wedge (y \vee z) \wedge (x \vee z) = x \wedge (y \vee z)$$

Therefore, $x \wedge y = x \wedge z$. From (2) and (4), we obtain that $y = z$. □

3.3 Canonical proportions and transitivity

In distributive lattices, it turns out that a basic form of AP, named *canonical proportion*, offers interesting properties, like being a basic factor of any AP. In non distributive lattices, these canonical proportions are also transitive, as proved below.

Proposition 5 *Let y and z be two elements of a lattice, the following AP, named canonical proportion, is true:*

$$y : y \vee z :: y \wedge z : z$$

Proof. The equalities of Proposition 2 can be easily checked using the absorption laws. □

Proposition 6 (Transitivity of canonical proportion) *If $x : y :: z : t$ and $z : t :: u : v$ are two canonical proportions in a lattice, then $x : y :: u : v$ is a canonical proportion.*

Proof. We know that $y = x \vee t$, $z = x \wedge t$, $t = z \vee v$ and $u = z \wedge v$. Hence, $v \leq t$ and $u = x \wedge t \wedge v = x \wedge v$. Similarly, $z \leq x$ and $y = x \vee z \vee v = x \vee v$. Then, $x : y :: u : v$ is a canonical proportion. □

Comment. In general, transitivity does not hold, as shown in the following example. In $\mathcal{S}$, $[0, 3] : \{3\} :: \{0\} : \varnothing$ and $\{0\} : \varnothing :: [0, 4] : \{4\}$ follow from Proposition 2, but $[0, 3] : \{3\} :: [0, 4] : \{4\}$ is wrong since $\{4\}$ is not equal to $([0, 4] \vee \{4\}) \wedge (\{3\} \vee \{4\}) = [3, 4]$.

3.4 Weak analogical proportions

As previously remarked in Proposition 3, the AP $x : y :: z : t$ implies $x \wedge t = y \wedge z$ and $x \vee t = y \vee z$ but *the reverse is generally false in a non distributive lattice*. Since the 8 equalities of Proposition 2 are quite restrictive to derive AP in such a lattice, we propose to introduce a weaker notion of analogical proportion.

Definition 4 *An element (x, y, z, t) of L^4 is a Weak Analogical Proportion (WAP) (or* Piaget Proportion*, see appendix of [13] and [14]) when $x \wedge t = y \wedge z$ and $x \vee t = y \vee z$. It is denoted $(x, t) \boxtimes (y, z)$.*

Comments. (a) Note that $\boxtimes$ is an equivalence relation between non ordered pairs of elements. (b) In a case of WAP, determinism is not true any more: in the lattice $\mathcal{S}$, one has $([2, 3], [8, 9]) \boxtimes ([2, 3], [7, 9])$.

4 Analogical proportions between formal concepts

The formal concepts that can be associated to a relation object-attribute can be organized into a lattice structure that is not distributive in general. Let us recall what are formal concepts.

4.1 Basics on formal concept analysis

Formal concept analysis starts with a binary relation R, called *formal context*, defined between a set $\mathcal{O}$ of objects and a set $\mathcal{A}$ of attributes (or Boolean properties). The notation $(o, a) \in R$ means that object o has the attribute a. $R^{\uparrow}(o) = \{a \in \mathcal{A} | (o, a) \in R\}$ is the set of attributes of object o and $R^{\downarrow}(a) = \{o \in \mathcal{O} | (o, a) \in R\}$ is the set of objects having attribute a.

Given a subset $\mathbf{a}$ of attributes, one can define the set $R^{\downarrow}(\mathbf{a}) = \{o \in \mathcal{O} | R^{\uparrow}(o) \supseteq \mathbf{a}\}$ which is the set of objects sharing all attributes in $\mathbf{a}$ (and having maybe some others) [6]. Similarly, for any subset $\mathbf{o}$ of objects, $R^{\uparrow}(\mathbf{o})$ is defined as $\{a \in \mathcal{A} | R^{\downarrow}(a) \supseteq \mathbf{o}\}$. Then a *formal concept* is defined as a pair $(\mathbf{o}, \mathbf{a})$ made of its *extension* $\mathbf{o}$ and its *intension* $\mathbf{a}$ such that $R^{\downarrow}(\mathbf{a}) = \mathbf{o}$ and $R^{\uparrow}(\mathbf{o}) = \mathbf{a}$. It can be also shown that formal concepts are maximal pairs $(\mathbf{o}, \mathbf{a})$ (in the sense of inclusion) such that $\mathbf{o} \times \mathbf{a} \subseteq R$.

Moreover, the set of all formal concepts is equipped with a partial order (denoted $\leq$) defined as: $(\mathbf{o}_1, \mathbf{a}_1) \leq (\mathbf{o}_2, \mathbf{a}_2)$ iff $\mathbf{o}_1 \subseteq \mathbf{o}_2$ (or, equivalently, $\mathbf{a}_2 \subseteq \mathbf{a}_1$), and forms a complete lattice, called the concept lattice of R.

Let us consider an example where R is a relation between eight attributes $a_1, \ldots, a_8$ and nine objects $o_1, \ldots, o_9$. The relation R is shown in Figure 1. There is a "×" in the cell corresponding to an object o and to an attribute a iff o has attribute a, in other words the "×"s describe the relation R (or context). There are 7 formal concepts in R. For instance, consider $\mathbf{o} = \{o_1, o_2, o_3, o_4, o_5\}$. Then $R^{\uparrow}(\mathbf{o}) = \{a_7, a_8\}$; likewise if $\mathbf{a} = \{7, 8\}$. Then $R^{\downarrow}(\mathbf{a}) = \{o_1, o_2, o_3, o_4, o_5\}$. The concept lattice of R is displayed on Figure 2.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
o_1							×	×
o_2							×	×
o_3					×	×	×	×
o_4					×	×	×	×
o_5					×	×	×	×
o_6					×	×		
o_7	×	×	×	×				
o_8			×	×				
o_9			×	×				

Figure 1: A context with 7 formal concepts

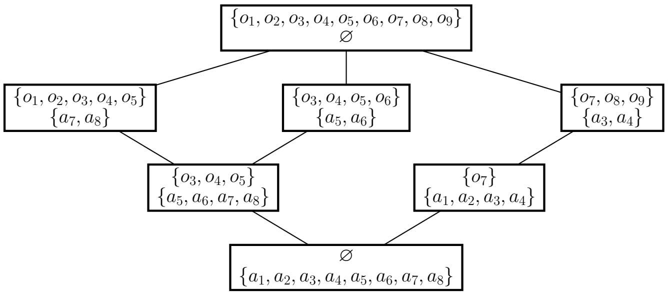


Figure 2: The formal concept lattice of R

4.2 Weak analogical proportions in concept lattices

Preliminary 1. Given two concepts $x = (\mathbf{o}_x, \mathbf{a}_x)$ and $y = (\mathbf{o}_y, \mathbf{a}_y)$ of a formal concept lattice, one has:

$$\mathbf{o}_x \cup \mathbf{o}_y \subseteq \mathbf{o}_{x \vee y}, \quad \mathbf{o}_x \cap \mathbf{o}_y = \mathbf{o}_{x \wedge y},$$
$$\mathbf{a}_x \cup \mathbf{a}_y \subseteq \mathbf{a}_{x \wedge y}, \quad \mathbf{a}_x \cap \mathbf{a}_y = \mathbf{a}_{x \vee y}\ .$$

Preliminary 2 Let $\mathbf{o}$ be a subset of $\mathcal{O}$, there exists at most one concept x such that $\mathbf{o}_x = \mathbf{o}$.

These properties are directly derivated from the definition of Concept Lattices and the Main Theorem of Formal Concepts [6, 4]. They allow for a quick demonstration of the following proposition.

Proposition 7 (Characterization of a WAP in a concept lattice) *Four elements $x = (\mathbf{o}_x, \mathbf{a}_x)$, $y = (\mathbf{o}_y, \mathbf{a}_y)$, $z = (\mathbf{o}_z, \mathbf{a}_z)$ and $t = (\mathbf{o}_t, \mathbf{a}_t)$ of a concept lattice are in WAP iff they are such that*

$$\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z \quad \textit{and} \quad \mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$$

Proof.

1. Suppose that $(x, t) \boxtimes (y, z)$. Then $x \vee t = y \vee z$, which implies, according to Preliminary 1, $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$. Similarly, we have $x \wedge t = y \wedge z$ that implies $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$.
2. Suppose that $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$. Then, $\mathbf{o}_{x \wedge t} = \mathbf{o}_x \cap \mathbf{o}_t$, by Preliminary 1. But there exists at most one concept for which the set of objects is $\mathbf{o}_x \cap \mathbf{o}_t$. Therefore, this concept is $x \wedge t$. The same reasoning shows that it is also $y \wedge z$. Therefore, $x \wedge t = y \wedge z$.

   We prove similarly that $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ implies $x \vee t = y \vee z$. Hence, $\mathbf{o}_x \cap \mathbf{o}_t = \mathbf{o}_y \cap \mathbf{o}_z$ and $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$ imply $(x, t) \boxtimes (y, z)$.

□

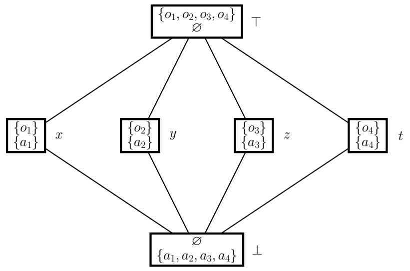
Comment. Suppose there are four distinct elements x, y, z and t of a concept lattice such that $|\mathbf{o}_x| = |\mathbf{o}_y| = |\mathbf{o}_z| = |\mathbf{o}_t| = 1$. These concepts are in WAP iff $\mathbf{a}_x \cap \mathbf{a}_t = \mathbf{a}_y \cap \mathbf{a}_z$, since the associated sets of objects are pairwise disjoint. However, this condition is not sufficient to obtain an AP between the four objects, considered as subsets of attributes. For example, in the following context, o_1, o_2, o_3 and o_4 are not in AP (due to attribute 5), while the concepts $x = (\{o_1\}, \{a_3, a_4\})$, $y = (\{o_2\}, \{a_1, a_3\})$, $z = (\{o_3\}, \{a_2, a_4\})$ and $t = (\{o_4\}, \{a_1, a_2, a_5\})$ are in WAP.

	a_1	a_2	a_3	a_4	a_5
o_1			×	×	
o_2	×		×		
o_3		×		×	
o_4	×	×			×

Let us consider some examples of lattices and their associated WAP. Firstly, we define a $(p, n)-$regular context as a $(n \times n)$ context with exactly p occurences of × on each line and on each column, and such that all lines are different, as well as all columns.

The following context corresponds to the $(1, 4)$-regular one. Additionally to the canonical proportions, several WAP can be derived: $(x, t) \boxtimes (y, z)$, $(x, y) \boxtimes (z, t)$ and $(x, t) \boxtimes (y, t)$. These WAP are degenerated cases since they link some concepts that are independent, regarding the associated context. In order to discard too simple or degenerated WAP, we introduce the following definition.

	a_1	a_2	a_3	a_4
o_1	×			
o_2		×		
o_3			×	
o_4				×



Definition 5 *A Full Weak Analogical Proportion (FWAP) is a WAP $(x, t) \boxtimes (y, z)$ where x, y, z and t are incomparable for $\leq$ and $(x, y) \boxtimes (z, t)$ and $(x, z) \boxtimes (y, t)$ are both false.*

Comment. Let us notice that trivial AP are not FWAP, like $x : y :: x : y$. Besides, if $(x, t) \boxtimes (y, z)$ is a FWAP, the object sets $\mathbf{o}_x$, $\mathbf{o}_y$, $\mathbf{o}_z$, $\mathbf{o}_t$ (resp. attribute sets $\mathbf{a}_x$, $\mathbf{a}_y$, $\mathbf{a}_z$, $\mathbf{a}_t$) are incomparable for $\subseteq$.

Proposition 8 *The $(2, 4)$-regular context, described in Figure 3, is the smallest (in terms of number of concepts as well as in size of context) concept lattice in which there is a FWAP.*

Proof. A context corresponding to a lattice with a FWAP must have at least four objects and four attributes. A systematic investigation of the $(4, 4)$ context gives the result. □

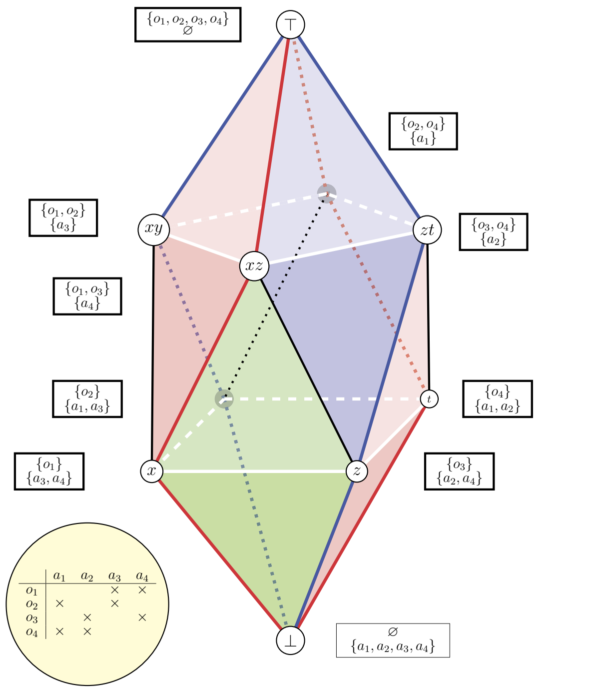
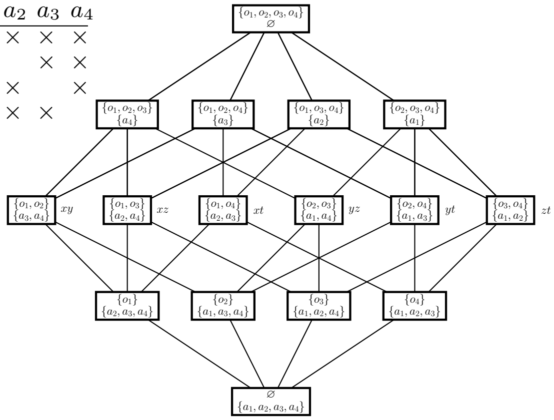


Figure 3: The $(2, 4)$-regular context and the associated concept lattice. The two FWAP's between concepts are indicated by white lined parallelograms.

The $(2,4)$-regular context actually has two FWAP's: $(x,t) \boxtimes (y,z)$ and $(xy,zt) \boxtimes (xz,yt)$. This lattice is not distributive, since $(\bot, x, t, xy, \top)$ is a pentagonal sublattice. Some (but not all) of the eight equalities of Proposition 2 are verified. For example: $(x \wedge y) \vee (x \wedge z) = x$, but $(x \vee y) \wedge (x \vee z) = \bot$. It displays 30 non canonical non full WAP, like $x : y :: xz : zt$.

As for the following $(3,4)$-regular context, it contains 16 concepts and exhibits three FWAP's $(xy,zt) \boxtimes (xz,yt) \boxtimes (xz,yz)$.

	a_1	a_2	a_3	a_4
o_1		×	×	×
o_2	×		×	×
o_3	×	×		×
o_4	×	×	×	



5 Proportional analogy

In this section, we are investigating whether we can establish a correspondance between a FWAP in a concept lattice and the notion of proportional analogy. In other words, having (x,t) and (y,z) four concepts in weak analogical proportion, we wonder whether there could be some expression "o_1 is to a_1 as o_2 is to a_2" that could be syntactically extracted from a FWAP in a concept lattice, with a semantic connexion to the ordinary meaning of an analogy, where o_1, o_2 are objects and a_1, a_2 are attributes, all concerned by the FWAP.

Let us start from the example "the fins are to the fish as the wings are to the bird", which can also be expressed as "the fins are the wings of the fish"[4]. How can this analogy be expressed in the framework of formal concept analysis ? Firstly, "fins" and "wings" are comparable, since they are organs useful to move in a fluid. They however are different since "fins" is related to a "fish" and "wings" to a "bird" (this last word does not appear in the phrase "the fins are the wings of the fish", but is implicit). Hence, it is natural to consider "wings" and "fins" as objects and "bird" and "fish" as attributes.

A fish and a bird are not only distinguished by the way they move. They don't breathe in the same manner, either. Oxygen in air is tranfered in the blood by the means of lungs, while in water it is transfered by gills. Also, the skin of a fish is protected by scales, while that of a bird is protected by feathers.

A formal context can be now extracted from this knowledge, namely:

	a_1	a_2	a_3	a_4	a_5
o_1: Fins	×		×		
o_2: Wings		×	×		
o_3: Gills	×			×	
o_4: Lungs		×		×	
o_5: Scales	×				×
o_6: Feathers		×			×

a_1: Part of a fish
a_2: Part of a bird
a_3: Mobility part
a_4: Breathing part
a_5: Covering part

[4] The first expression is a plain analogy, the second is a metaphorical expression of the same analogy

We have now three similar analogies: "the fins are to the fish as the wings are to the bird", "the scales are to the fish as the feathers are to the bird" and "the gills are to the fish as the the lungs are to the bird". Let us draw the corresponding lattice (Figure 4).

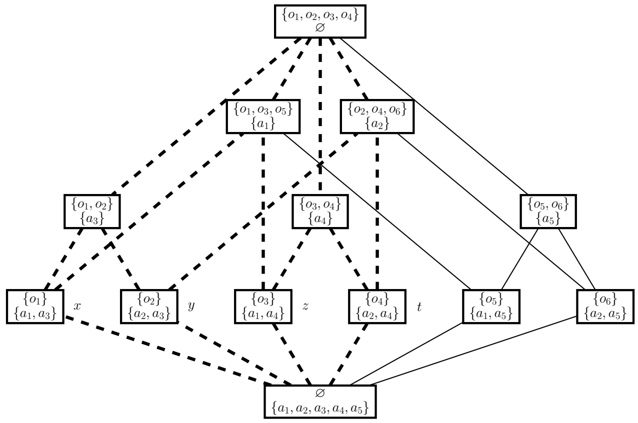


Figure 4

What is the common graphic structure to the three analogies "o_1 is to a_1 as o_2 is to a_2", "o_3 is to a_1 as o_4 is to a_2" and "o_5 is to a_1 as o_6 is to a_2"? Consider the sublattice in dashed lines in which one has the analogy "o_1 is to a_1 as o_2 is to a_2". The basic remarks are that:

- The couples of concepts $\big((\{o_1\},\{a_1,a_3\}),(\{o_4\},\{a_2,a_4\})\big)$ and $\big((\{o_2\},\{a_2,a_3\}),(\{o_3\},\{a_1,a_4\})\big)$ are in FWAP,
- $\{a_1\} = \{a_1,a_3\} \backslash \{a_2,a_3\}$,
- $\{a_2\} = \{a_2,a_3\} \backslash \{a_1,a_3\}$.

And these remarks can be transposed exactly to the other analogies. Hence, we can tentatively say "o_1 is to a_1 as o_2 is to a_2" when

- $\{o_1\}$ and $\{o_2\}$ are the sets of objects of two concepts x and y of a FWAP $(x,t) \boxtimes (y,z)$,
- $\{a_1\} = \mathbf{a}_x \backslash \mathbf{a}_y$.
- $\{a_2\} = \mathbf{a}_y \backslash \mathbf{a}_x$.

However, this definition has to be made precise to include more complex cases.

We denote an analogy "o_1 is to a_1 as o_2 is to a_2" such as:

$$o_1 \updownarrow a_1 \updownarrow\!\updownarrow o_2 \updownarrow a_2$$

and we note that the $\updownarrow\!\updownarrow$ operator is commutative, but that permuting the extreme or the means makes no sense. This leads to the following definition.

Definition 6 (Proportional analogy) *A proportional analogy is a relation between objects and attributes, derived from a FWAP* $(x,t) \boxtimes (y,z)$ *between concepts in the following manner:*

$$(\mathbf{o}_x \backslash \mathbf{o}_y) \updownarrow (\mathbf{a}_x \backslash \mathbf{a}_y) \updownarrow\!\updownarrow (\mathbf{o}_y \backslash \mathbf{o}_x) \updownarrow (\mathbf{a}_y \backslash \mathbf{a}_x)\,.$$

This definition makes clear that in order to build a proportional analogy on the basis of two formal concepts x and y, we need a formal context which there exist other concepts z and t involving other objects and attributes, in such a way that (x,y,z,t) makes a FWAP, i.e., a form of analogical proportion not trivialized by dependence or independence relations.

Example. Let us illustrate the above definition by the following context

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
o_1	×				×		×	
o_2		×				×	×	
o_3			×		×			×
o_4				×		×		×
o_5					×			
o_6						×		
o_7							×	
o_8								×

with this interpretation of the objects and attributes:

o_1 :	Fins	a_1 :	Part of a Whale
o_2 :	Wings	a_2 :	Part of a Bat
o_3 :	Scales	a_3 :	Part of a Snake
o_4 :	Feathers	a_4 :	Part of a Deinonychus
o_5 :	Gills	a_5 :	Part of a Fish
o_6 :	Beak	a_6 :	Part of a Bird
o_7 :	Hooves	a_7 :	Cymbulia peroni
o_8 :	Thick fur	a_8 :	Chimera

A Cymbulia peroni is a shell-less pteropod with a large hoof-like pseudoconch. Its fins look like wings. For enriching the formal context of the example, we have also introduced an imaginary animal, known as Chimera.

We can construct the corresponding lattice (see Figure 5) and extract the FWAP $(x,t) \boxtimes (y,z)$ with $x = (\{o_1\},\{a_1,a_5,a_7,a_9\})$, $y = (\{o_2\},\{a_2,a_6,a_7,a_9\})$, $z = (\{o_3\},\{a_3,a_5,a_8,a_9\})$ and $t = (\{o_4\},\{a_4,a_6,a_8,a_9\})$.

According to our definition, the relation "$\{o_1\} \updownarrow \{a_1,a_5\} \updownarrow\!\updownarrow \{o_2\} \updownarrow \{a_2,a_6\}$" is a formal analogy, i.e. "the fins are to the fish and the whale as the wings are to the bird and the bat". The symmetry property of the WAP gives other proportional analogies as "$\{o_3\} \updownarrow \{a_3,a_5\} \updownarrow\!\updownarrow \{o_4\} \updownarrow \{a_4,a_6\}$".

Similarly, from Figure 5 and the FWAP $(x',t') \boxtimes (y',z')$ where $x' = (\{o_1,o_3,o_5\},\{a_5\})$, $y' = (\{o_3,o_4,o_8\},\{a_8\}))$, $z' = (\{o_1,o_2,o_8\},\{a_7\})$ and $t' = (\{o_2,o_4,o_6\},\{a_6\})$, we obtain among other proportional analogies "$\{o_1,o_5\} \updownarrow \{a_5\} \updownarrow\!\updownarrow \{o_4,o_8\} \updownarrow \{a_8\}$".

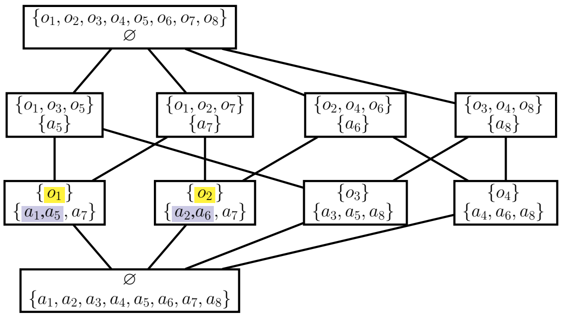


Figure 5: Construction of the analogy $\{a\} \updownarrow \{1,5\} \updownarrow\!\updownarrow \{b\} \updownarrow \{2,6\}$ or "the fins are to the fish as the wings are to the bird" from the lattice corresponding to the example context.

Starting with the notion of analogical proportion defined in a lattice structure, and choosing the most compatible form with a non-ditributive lattice, we have applied this definition to formal concept lattices and focused on the meaningful analogical proportions in such a setting, in order to finally define the notion of proportional analogy, by sort-of projecting such analogical proportions on the objects and attributes of a formal context. The properties of proportional analogies remain to be investigated.

6 Conclusion

It is only recently that analogical proportions have been formalized in various settings. When applied to classification, analogical proportions hold between examples described by vectors of Boolean features, i.e. between items of the same nature. However, statements having the form of analogical proportions often involve both objects and attributes. This has led us to the question of relating formal concept analysis and analogical proportions. In this paper, we have proposed a definition of analogical proportion which is suitable for a non-distributive lattice structure as the one underlying formal concepts. We have shown that it was then possible to extract, from a concept lattice, proportional analogies pairing two (object, attribute)-pairs. It thus provides a new view of the idea of analogy that is worth investigating. Besides, analogical proportions when defined on a unique domain have been proved to be fruitful in classification tasks [10, 15, 3], while formal concept analysis underlies data mining, which suggests that in the long range the ideas presented here may have some impact in machine learning.

REFERENCES

[1] Aristotle, *Poetics, chap. XXI.* transl. S. H. Butcher.

[2] N. Barbot, L. Miclet, and H. Prade, 'Analogical proportions and the factorization of information in distributive lattices', in *CEUR Workshop Proc. 10th Inter. Conf. Concept Lattices & Applications (CLA'13), La Rochelle*, eds., M. Ojeda-Aciego and J. Outrata, pp. 175–186, (2013).

[3] M. Bounhas, H. Prade, and G. Richard, 'Analogical classification: A new way to deal with examples', in *Proc. 21th Eur. Conf. on Artificial Intelligence, Praha, Aug. 18-22*, (2014).

[4] R. Bělohlávek, 'Introduction to formal concept analysis', Technical report, University of Palacký, (2008).

[5] R. Faure and E. Heurgon, *Structures ordonnées et algèbres de Boole*, Gauthier-Villars, 1970.

[6] B. Ganter and R. Wille, *Formal Concept Analysis. Mathematical Foundations*, Springer-Verlag, 1999.

[7] D. Gentner, K. J. Holyoak, and B. N. Kokinov, eds., *The Analogical Mind: Perspectives from Cognitive Science*, Cognitive Science, and Philosophy, MIT Press, Cambridge, MA, 2001.

[8] M. Hesse, 'On defining analogy', *Proceedings of the Aristotelian Society*, **60**, 79–100, (1959).

[9] Y. Lepage, 'Analogy and formal languages', *Electr. Notes Theor. Comput. Sci.*, **53**, (2001).

[10] L. Miclet, S. Bayoudh, and A. Delhay, 'Analogical dissimilarity: definition, algorithms and two experiments in machine learning', *JAIR, 32*, 793–824, (2008).

[11] L. Miclet and H. Prade, 'Handling analogical proportions in classical logic and fuzzy logics settings', in *Proc. 10th Eur. Conf. on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09),Verona*, pp. 638–650. Springer, LNCS 5590, (2009).

[12] L. Miclet, H. Prade, and D. Guennec, 'Looking for analogical proportions in a formal concept analysis setting', in *Proc. 8th Inter. Conf. on Concept Lattices & Applications (CLA'11)*, eds., A. Napoli and V. Vychodil, pp. 295 – 307. INRIA, Nancy, (2011).

[13] J. Piaget, *Essai sur les transformations des opérations logiques: les 256 opérations ternaires de la logique bivalente des propositions*, Presses Universitaires de France, Paris, 1952.

[14] J. Piaget, *Logic and Psychology*, Manchester Univ. Press, 1953.

[15] H. Prade and G. Richard, eds., *Computational Approaches to Analogical Reasoning: Current Trends*, volume 548 of *Studies in Computational Intelligence*, Springer, 2014.

[16] N. Stroppa, *Définitions et caractérisations de modèles à base d'analogies pour l'apprentissage automatique des langues naturelles*, Ph.D. dissertation, 2005.

[17] N. Stroppa and F. Yvon, 'Du quatrième de proportion comme principe inductif : une proposition et son application à l'apprentissage de la morphologie', *Traitement Automatique des Langues*, **47**(2), 1–27, (2006).

[18] N. Stroppa and F. Yvon, 'Formal models of analogical proportions', Technical Report 2006D008, École Nationale Supérieure des Télécommunications, Paris, (2006).