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Belief merging within fragments of propositional logic

Nadia Creignou[1] and **Odile Papini**[1] and **Stefan Rümmele**[2] and **Stefan Woltran**[2]

Abstract. Recently, belief change within the framework of fragments of propositional logic has gained increasing attention. Previous works focused on belief contraction and belief revision on the Horn fragment. However, the problem of belief merging within fragments of propositional logic has been neglected so far. This paper presents a general approach to define new merging operators derived from existing ones such that the result of merging remains in the fragment under consideration. Our approach is not limited to the case of Horn fragment but applicable to any fragment of propositional logic characterized by a closure property on the sets of models of its formulæ. We study the logical properties of the proposed operators in terms of satisfaction of merging postulates, considering in particular distance-based merging operators for Horn and Krom fragments.

1 Introduction

Belief merging consists in achieving a synthesis between pieces of information provided by different sources. Although these sources are individually consistent, they may mutually conflict. The aim of merging is to provide a consistent set of information, making maximum use of the information provided by the sources while not favoring any of them. Belief merging is an important issue in many fields of Artificial Intelligence (AI) [3] and symbolic approaches to multi-source fusion gave rise to increasing interest within the AI community since the 1990s [2, 5, 12, 15, 16]. One of today's major approaches is the problem of merging under (integrity) constraints in order to generalize both merging (without constraints) and revision (of old information by a new piece of information). For the latter the constraints then play the role of the new piece of information. Postulates characterizing the rational behavior of such merging operators, known as IC postulates, have been proposed by Revesz [15] and improved by Konieczny and Pino Pérez [10] in the same spirit as the seminal AGM [1] postulates for revision. Concrete merging operators have been proposed according to either semantic (model-based) or syntactic (formula-based) points of view in a classical logic setting. We focus here on the model-based approach of distance-based merging operators [9, 10, 16]. These operators are parametrized by a distance which represents the closeness between interpretations and an aggregation function which captures the merging strategy and takes the origin of beliefs into account.

Belief change operations within the framework of fragments of classical logic constitute a vivid research branch. In particular, contraction [4, 8, 18] and revision [7, 14, 19] have been thoroughly analyzed in the literature. The motivation for such a research is twofold:

- In many applications, the language is restricted a priori. For instance, a rule-based formalization of expert's knowledge is much easier to handle for standard users. In case users want to revise or merge some sets of rules, they indeed expect that the outcome is still in the easy-to-read format they are used to.
- Many fragments of propositional logic allow for efficient reasoning methods. Suppose an agent has to make a decision according to a group of experts' beliefs. This should be done efficiently, therefore the expert's beliefs are stored as formulæ known to be in a tractable class. For making a decision, it is desired that the result of the change operation yields a set of formulæ in the same fragment. Hence, the agent still can use the dedicated solving method she is equipped with for this fragment.

Most of previous work has focused on the Horn fragment except [6] that studied revision in any fragment of propositional logic. However, as far as we know, the problem of *belief merging* within *fragments of propositional logic* has been neglected so far.

The main obstacle hereby is that for a language fragment $\mathcal{L}'$, given n belief bases $K_1, \ldots, K_n \in 2^{\mathcal{L}'}$ and a constraint $\mu \in \mathcal{L}'$, there is no guarantee that the outcome of the merging, $\Delta_\mu(\{K_1, \ldots, K_n\})$, remains in $\mathcal{L}'$ as well. Let for example, $K_1 = \{a\}$, $K_2 = \{b\}$ and $\mu = \neg a \vee \neg b$ be two sets of formulæ and a formula expressed in the Horn fragment. Merging with typical distance-based operator proposed in [10] does not remain in the Horn language fragment since the result of merging is equivalent to $(a \vee b) \wedge (\neg a \vee \neg b)$, which is not equivalent to any Horn formula (see [17]).

We propose the concept of *refinement* to overcome these problems. Refinements have been proposed for revision in [6] and capture the intuition of adapting a given operator (defined for full classical logic) in order to become applicable within a fragment. The basic properties of a refinement are thus (i) to guarantee the result of the change operation to be in the same fragment as the belief change scenario given and (ii) to keep the behavior of the original operator unchanged in case it delivers a result which already fits in the fragment.

Refinements are interesting from different points of view. Several fragments can be treated in a uniform way and a general characterization of refinements is provided for any fragment. Defining and studying refinements of merging operators is not a straightforward extension of the revision case. It is more complex due to the nature of the merging operators. Even if the constraints play the role of the new piece of information in revision, model-based merging deals with multi-sets of models. Moreover applying this approach to different distance-based merging operators, each parameterized by a distance and an aggregation function, reveals that all the different parameters matter, thus showing a rich variety of behaviors for refined merging operators.

The main contributions of this paper are the following:

- We propose to adapt known belief merging operators to make them applicable in fragments of propositional logic. We provide natural criteria which refined operators should satisfy. We charac-

[1] Aix Marseille Université, CNRS, email: name.surname@univ-amu.fr
[2] Vienna University of Technology, email: surname@dbai.tuwien.ac.at

terize refined operators in a constructive way.

- This characterization allows us to study their properties in terms of the IC postulates [10]. On one hand we prove that the basic postulates (IC0–IC3) are preserved for any refinement for any fragment. On the other hand we show that the situation is more complex for the remaining postulates. We provide detailed results for the Horn and the Krom fragment in terms of two kinds of distance-based merging operators and three approaches for refinements.

2 Preliminaries

Propositional Logic. We consider $\mathcal{L}$ as the language of propositional logic over some fixed alphabet $\mathcal{U}$ of propositional atoms. A clause is a disjunction of literals. A clause is called *Horn* if at most one of its literals is positive; and *Krom* if it consists of at most two literals. We identify the following subsets of $\mathcal{L}$: $\mathcal{L}_{Horn}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of Horn clauses, and $\mathcal{L}_{Krom}$ is the set of all formulæ in $\mathcal{L}$ being conjunctions of Krom clauses. In what follows we sometimes just talk about arbitrary fragments $\mathcal{L}' \subseteq \mathcal{L}$. Hereby, we tacitly assume that any such fragment $\mathcal{L}' \subseteq \mathcal{L}$ contains at least the formula $\top$.

An interpretation is represented either by a set $\omega \subseteq \mathcal{U}$ of atoms (corresponding to the variables set to true) or by its corresponding characteristic bit-vector of length $|\mathcal{U}|$. For instance if we consider $\mathcal{U} = \{x_1, \ldots, x_6\}$, the interpretation $x_1 = x_3 = x_6 = 1$ and $x_2 = x_4 = x_5 = 0$ will be represented either by $\{x_1, x_3, x_6\}$ or by $(1, 0, 1, 0, 0, 1)$. As usual, if an interpretation ω satisfies a formula ϕ, we call ω a model of ϕ. By $\mathrm{Mod}(\phi)$ we denote the set of all models (over $\mathcal{U}$) of ϕ. Moreover, $\psi \models \phi$ if $\mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\phi)$ and $\psi \equiv \phi$ (ϕ and ψ are equivalent) if $\mathrm{Mod}(\psi) = \mathrm{Mod}(\phi)$.

A *base* K is a finite set of propositional formulæ $\{\varphi_1, \ldots, \varphi_n\}$. We shall often identify K via $\bigwedge K$, the conjunction of formulæ of K, i.e., $\bigwedge K = \varphi_1 \wedge \cdots \wedge \varphi_n$. Thus, a base K is said to be consistent if $\bigwedge K$ is consistent, $\mathrm{Mod}(K)$ is a shortcut for $\mathrm{Mod}(\bigwedge K)$, $K \models \phi$ stands for $\bigwedge K \models \phi$, etc. Given $\mathcal{L}' \subseteq \mathcal{L}$ we denote by $\mathcal{K}_{\mathcal{L}'}$ the set of bases restricted to formulæ from $\mathcal{L}'$. For fragments $\mathcal{L}' \subseteq \mathcal{L}$, we also use $T_{\mathcal{L}'}(K) = \{\phi \in \mathcal{L}' \mid K \models \phi\}$.

A *profile* E is a non-empty finite multiset of consistent bases $E = \{K_1, \ldots, K_n\}$ and represents a group of n agents having different beliefs. Given $\mathcal{L}' \subseteq \mathcal{L}$, we denote by $\mathcal{E}_{\mathcal{L}'}$ the set of profiles restricted to the use of formulæ from $\mathcal{L}'$. We denote $\bigwedge K_1 \wedge \ldots \wedge \bigwedge K_n$ by $\bigwedge E$. The profile is said to be consistent if $\bigwedge E$ is consistent. By abuse of notation we write $K \sqcup E$ to denote the multi-set union $\{K\} \sqcup E$. The multi-set consisting of the sets of models of the bases in a profile is denoted $\mathcal{M}od(E) = \{\mathrm{Mod}(K_1), \ldots, \mathrm{Mod}(K_n)\}$. Two profiles E_1 and E_2 are equivalent, denoted by $E_1 \equiv E_2$ if $\mathcal{M}od(E_1) = \mathcal{M}od(E_2)$. Finally, for a set of interpretations $\mathcal{M}$ and a profile E we define $\#(\mathcal{M}, E) = |\{i : \mathcal{M} \cap \mathrm{Mod}(K_i) \neq \emptyset\}|$.

Characterizable Fragments of Propositional Logic. Let $\mathcal{B}$ denote the set of all Boolean functions $\beta\colon \{0,1\}^k \to \{0,1\}$ that have the following two properties

- *symmetry*, i.e., for all permutations σ, $\beta(x_1, \ldots, x_k) = \beta(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ and
- 0- and 1-*reproduction*, i.e., for all $x \in \{0,1\}$, $\beta(x, \ldots, x) = x$.

Examples are the binary AND function denoted by $\wedge$ or the ternary MAJORITY function, $\mathrm{maj}_3(x, y, z) = 1$ if at least two of the variables x, y, and z are set to 1. We extend Boolean functions to interpretations by applying coordinate-wise the original function (recall that we consider interpretations also as bit-vectors). So, if $M_1, \ldots, M_k \in \{0,1\}^n$, then $\beta(M_1, \ldots, M_k)$ is defined by $(\beta(M_1[1], \ldots, M_k[1]), \ldots, \beta(M_1[n], \ldots, M_k[n]))$, where $M[i]$ is the i-th coordinate of the interpretation M.

Definition 1. *Given a set $\mathcal{M} \subseteq 2^{\mathcal{U}}$ of interpretations and $\beta \in \mathcal{B}$, we define $Cl_\beta(\mathcal{M})$, the* closure *of $\mathcal{M}$ under β, as the smallest set of interpretations that contains $\mathcal{M}$ and that is closed under β, i.e., if $M_1, \ldots, M_k \in Cl_\beta(\mathcal{M})$, then also $\beta(M_1, \ldots, M_k) \in Cl_\beta(\mathcal{M})$.*

Let us mention some easy properties of such a closure: (i) monotonicity; (ii) if $|\mathcal{M}| = 1$, then $Cl_\beta(\mathcal{M}) = \mathcal{M}$; (iii) $Cl_\beta(\emptyset) = \emptyset$.

Definition 2. *Let $\beta \in \mathcal{B}$. A set $\mathcal{L}' \subseteq \mathcal{L}$ of propositional formulæ is a β-*fragment *(or* characterizable fragment*) if:*

1. *for all $\psi \in \mathcal{L}'$, $\mathrm{Mod}(\psi) = Cl_\beta(\mathrm{Mod}(\psi))$*
2. *for all $\mathcal{M} \subseteq 2^{\mathcal{U}}$ with $\mathcal{M} = Cl_\beta(\mathcal{M})$ there exists a $\psi \in \mathcal{L}'$ with $\mathrm{Mod}(\psi) = \mathcal{M}$*
3. *if $\phi, \psi \in \mathcal{L}'$ then $\phi \wedge \psi \in \mathcal{L}'$.*

It is well-known that $\mathcal{L}_{Horn}$ is an $\wedge$-fragment and $\mathcal{L}_{Krom}$ is a maj_3-fragment (see e.g. [17]).

Logical Merging Operators. Belief merging aims at combining several pieces of information coming from different sources. Merging operators we consider are functions from the set of profiles and the set of propositional formulæ to the set of bases, i.e., $\Delta\colon \mathcal{E}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}}$. For $E \in \mathcal{E}_{\mathcal{L}}$ and $\mu \in \mathcal{L}$ we will write $\Delta_\mu(E)$ instead of $\Delta(E, \mu)$; the formula μ is referred to as the *integrity constraint* (IC) and restricts the result of the merging.

As for belief revision some logical properties that one could expect from any reasonable merging operator have been stated. See [10] for a detailed discussion. Intuitively $\Delta_\mu(E)$ is the "closest" belief base to the profile E satisfying the integrity constraint μ. This is what the following postulates try to capture.

(IC0) $\Delta_\mu(E) \models \mu$
(IC1) If μ is consistent, then $\Delta_\mu(E)$ is consistent
(IC2) If $\bigwedge E$ is consistent with μ, then $\Delta_\mu(E) = \bigwedge E \wedge \mu$
(IC3) If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$, then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$
(IC4) If $K_1 \models \mu$ and $K_2 \models \mu$, then $\Delta_\mu(\{K_1, K_2\}) \wedge K_1$ is consistent if and only if $\Delta_\mu(\{K_1, K_2\}) \wedge K_2$ is consistent
(IC5) $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2) \models \Delta_\mu(E_1 \sqcup E_2)$
(IC6) If $\Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$ is consistent, then $\Delta_\mu(E_1 \sqcup E_2) \models \Delta_\mu(E_1) \wedge \Delta_\mu(E_2)$
(IC7) $\Delta_{\mu_1}(E) \wedge \mu_2 \models \Delta_{\mu_1 \wedge \mu_2}(E)$
(IC8) If $\Delta_{\mu_1}(E) \wedge \mu_2$ is consistent, then $\Delta_{\mu_1 \wedge \mu_2}(E) \models \Delta_{\mu_1}(E)$

Similarly to belief revision, a representation theorem [10] shows that a merging operator corresponds to a family of total preorders over interpretations satisfying certain conditions. More formally, for $E \in \mathcal{E}_{\mathcal{L}}$, $\mu \in \mathcal{L}$ and $\leq_E$ a total preorder over interpretations, a model-based operator is defined by $\mathrm{Mod}(\Delta_\mu(E)) = min(\mathrm{Mod}(\mu), \leq_E)$. The model-based merging operators select interpretations that are the "closest" to the original belief bases.

Distance-based operators where the notion of closeness stems from the definition of a distance (or a pseudo-distance[3]) between interpretations and from an aggregation function have been proposed in [10, 11]. More formally, let $E = \{K_1, \ldots, K_n\} \in \mathcal{E}_{\mathcal{L}}$, $\mu \in \mathcal{L}$, d be a distance and f be an aggregation function, we consider the

[3] Let $\omega, \omega' \in \mathcal{W}$, a pseudo-distance is such that $d(\omega, \omega') = d(\omega', \omega)$ and $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$.

family of $\Delta_\mu^{d,f}$ merging operators defined by $\mathrm{Mod}(\Delta_\mu^{d,f}(E)) = min(\mathrm{Mod}(\mu), \leq_E)$ where $\leq_E$ is a total preorder over the set $2^{\mathcal{U}}$ of interpretations defined as follows:

- $d(\omega, K_i) = min_{\omega' \models K_i} d(\omega, \omega')$,
- $d(\omega, E) = f(d(\omega, K_1), \dots, d(\omega, K_n))$, and
- $\omega \leq_E \omega'$ if $d(\omega, E) \leq d(\omega', E)$.

Definition 3. *A* counting distance *between interpretations is a function $d : 2^{\mathcal{U}} \times 2^{\mathcal{U}} \to \mathbb{R}^+$ defined for every pair of interpretations (ω, ω') by $d(\omega, \omega') = g(|(\omega \setminus \omega') \cup (\omega' \setminus \omega)|)$, where $g : \mathbb{N} \to \mathbb{R}^+$ is a nondecreasing function such that $g(n) = 0$ if and only if $n = 0$. If $g(n) = g(1)$ for every $n \neq 0$, we call d a* drastic distance *and denote it via d_D. If $g(n) = n$ for all n, we call d the* Hamming distance *and denote it via d_H. If for every interpretations w, w' and w'' we have $d(w, w') \leq d(w, w'') + d(w'', w')$, then we say that the distance d satisfies the triangular inequality.*

Observe that a counting distance is indeed a pseudo-distance, and both, the Hamming distance and the drastic distance satisfy the triangular inequality.

As aggregation functions, we consider here Σ, the sum aggregation function, and the aggregation function GMax defined as follows. Let $E = \{K_1, \dots, K_n\} \in \mathcal{E}_{\mathcal{L}}$ and ω, ω' be two interpretations. Let $(d_1^\omega, \dots, d_n^\omega)$, where $d_j^\omega = d_H(\omega, K_j)$, be the vector of distances between ω and the n belief bases in E. Let L_ω^E be the vector obtained from $(d_1^\omega, \dots, d_n^\omega)$ by ranking it in decreasing order. The aggregation function GMax is defined by $\mathrm{GMax}(d_1^\omega, \dots, d_n^\omega) = L_\omega^E$, with $\mathrm{GMax}(d_1^\omega, \dots, d_n^\omega) \leq \mathrm{GMax}(d_1^{\omega'}, \dots, d_n^{\omega'})$ if $L_\omega^E \leq_{lex} L_{\omega'}^E$, where $\leq_{lex}$ denotes the lexicographical ordering.

In this paper we focus on the $\Delta^{d,\Sigma}$ and $\Delta^{d,\mathrm{GMax}}$ operators where d is an arbitrary counting distance. These operators are known to satisfy the postulates (IC0)–(IC8), as shown in [9] generalizing more specific results from [10, 13]. Note that these two operators coincide for the drastic distance. Finally, we define certain concepts for merging operators and fragments.

Definition 4. A basic *(merging) operator for $\mathcal{L}' \subseteq \mathcal{L}$ is any function $\Delta : \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ satisfying $\mathrm{Mod}(\Delta_\mu(\{\{\top\}\})) = \mathrm{Mod}(\mu)$ for each $\mu \in \mathcal{L}'$. We say that Δ* satisfies *an* (IC) *postulate* (IC_i) *($i \in \{0, \dots, 8\}$)* in $\mathcal{L}'$ *if the respective postulate holds when restricted to formulæ from $\mathcal{L}'$.*

3 Refined Operators

Let us reconsider the example from Section 1 to illustrate the problem of standard operators when applied within a β-fragment.

Example 1. *Let $\mathcal{U} = \{a, b\}$, $E = \{K_1, K_2\} \in \mathcal{E}_{\mathcal{L}_{Horn}}$ and $\mu \in \mathcal{L}_{Horn}$ such that $\mathrm{Mod}(K_1) = \{\{a\}, \{a, b\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}, \{a, b\}\}$, and $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}\}$. Consider the distance-based merging operators, $\Delta^{d_H,\Sigma}$ and $\Delta^{d_H,\mathrm{GMax}}$. The following table gives the distances between the interpretations of μ and the belief bases, and the result of the aggregation functions Σ and* GMax.

	K_1	K_2	Σ	GMax
$\emptyset$	1	1	2	$(1, 1)$
$\{a\}$	0	1	1	$(1, 0)$
$\{b\}$	1	0	1	$(1, 0)$

Hence, we have $\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)) = \mathrm{Mod}(\Delta_\mu^{d_H,\mathrm{GMax}}(E)) = \{\{a\}, \{b\}\}$. Thus, for instance, we can give $\phi = (a \vee b) \wedge (\neg a \vee \neg b)$ as a result of the merging for both operators. However, there is no $\psi \in \mathcal{L}_{Horn}$ with $\mathrm{Mod}(\psi) = \{\{a\}, \{b\}\}$ (each $\psi \in \mathcal{L}_{Horn}$ satisfies the following closure property in terms of its set of models: for every $I, J \in \mathrm{Mod}(\psi)$, also $I \cap J \in \mathrm{Mod}(\psi)$)). Thus, the result of the operator has to be "refined", such that it fits into the Horn fragment. On the other hand, it holds that $\mu \in \mathcal{L}_{Krom}$, $E \in \mathcal{E}_{\mathcal{L}_{Krom}}$ and also the result ϕ is in Krom. This shows that different fragments behave differently on certain instances. Nonetheless, we aim for a uniform approach for refining merging operators.

We are interested in the following: Given a known merging operator Δ and a fragment $\mathcal{L}'$ of propositional logic, how can we adapt Δ to a new merging operator $\Delta^\star$ such that, for each $E \in \mathcal{E}_{\mathcal{L}'}$ and $\mu \in \mathcal{L}'$, $\Delta_\mu^\star(E) \in \mathcal{K}_{\mathcal{L}'}$? Let us define a few natural desiderata for $\Delta^\star$ inspired by the work on belief revision. See [6] for a discussion.

Definition 5. *Let $\mathcal{L}'$ be a fragment of classical logic and Δ a merging operator. We call an operator $\Delta^\star \colon \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ a Δ-*refinement *for $\mathcal{L}'$ if it satisfies the following properties, for each $E, E_1, E_2 \in \mathcal{E}_{\mathcal{L}'}$ and $\mu, \mu_1, \mu_2 \in \mathcal{L}'$.*

1. *consistency: $\Delta_\mu(E)$ is consistent if and only if $\Delta_\mu^\star(E)$ is consistent*
2. *equivalence: if $E_1 \equiv E_2$ and $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$ then $\Delta_{\mu_1}^\star(E_1) \equiv \Delta_{\mu_2}^\star(E_2)$*
3. *containment: $T_{\mathcal{L}'}(\Delta_\mu(E)) \subseteq T_{\mathcal{L}'}(\Delta_\mu^\star(E))$*
4. *invariance: If $\Delta_\mu(E) \in \mathcal{K}_{\langle \mathcal{L}' \rangle}$, then $T_{\mathcal{L}'}(\Delta_\mu^\star(E)) \subseteq T_{\mathcal{L}'}(\Delta_\mu(E))$, where $\langle \mathcal{L}' \rangle$ denotes the set of formulæ in $\mathcal{L}$ for which there exists an equivalent formula in $\mathcal{L}'$.*

One can show that a Δ-refinement $\Delta^\star$ for a β-fragment satisfies the properties: (i) $\mathrm{Mod}(\Delta_\mu^\star(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$ and (ii) $\mathrm{Mod}(\Delta_\mu^\star(E)) = \mathrm{Mod}(\Delta_\mu(E))$ in case $\mathrm{Mod}(\Delta_\mu(E))$ is closed under β. This motivates the following candidates for such refinements.

Definition 6. *Let Δ be a merging operator and $\beta \in \mathcal{B}$. We define the* Cl_β-based *refined operator Δ^{Cl_β} as:*

$$\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = Cl_\beta(\mathcal{M}).$$

where $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E))$.

We define the Min-based *refined operator Δ^{Min} as:*

$$\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)) = \begin{cases} \mathcal{M} & \text{if } Cl_\beta(\mathcal{M}) = \mathcal{M}, \\ \{\mathrm{Min}(\mathcal{M})\} & \text{otherwise,} \end{cases}$$

where Min *is a function that selects a single interpretation from a set of interpretations with respect to a given and fixed order.*

We define the Min/Cl_β-based *refined operator $\Delta^{\mathrm{Min}/Cl_\beta}$ as:*

$$\Delta_\mu^{\mathrm{Min}/Cl_\beta}(E) = \begin{cases} \Delta_\mu^{\mathrm{Min}}(E) & \text{if } \#(\mathcal{M}, E) = 0 \\ \Delta_\mu^{Cl_\beta}(E) & \text{otherwise.} \end{cases}$$

Proposition 1. *For any merging operator $\Delta : \mathcal{E}_{\mathcal{L}} \times \mathcal{L} \to \mathcal{K}_{\mathcal{L}}$, $\beta \in \mathcal{B}$ and $\mathcal{L}' \subseteq \mathcal{L}$ a β-fragment, the operators Δ^{Cl_β}, Δ^{Min} and $\Delta_\mu^{\mathrm{Min}/Cl_\beta}$ are Δ-refinements for $\mathcal{L}'$.*

Example 2. *Consider the profile E, the integrity constraint μ given in Example 1, the distance-based merging operator $\Delta^{d_H,\Sigma}$, and let β be the binary AND function. Let us have the following order over the set of interpretations on $\{a, b\}$: $\emptyset < \{a\} < \{b\} < \{a, b\}$. The result of merging is $\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)) = \{\{a\}, \{b\}\}$. The* Min-*based $\Delta^{d_H,\Sigma}$-refined operator, denoted by Δ^{Min}, is such that $\mathrm{Mod}(\Delta_\mu^{\mathrm{Min}}(E)) = \{\{a\}\}$. The Cl_β-based $\Delta^{d_H,\Sigma}$-refined operator, denoted by Δ^{Cl_β}, is such that $\mathrm{Mod}(\Delta_\mu^{Cl_\beta}(E)) = \{\{a\}, \{b\}, \emptyset\}$. The same result is achieved by the the Min/Cl_β-based $\Delta^{d_H,\Sigma}$-refined operator since $\#(\mathrm{Mod}(\Delta_\mu^{d_H,\Sigma}(E)), E) = 2$.*

In what follows we show how to capture not only a particular refined operator but characterize the class of *all* refined operators.

Definition 7. *Given $\beta \in \mathcal{B}$, we define a β-mapping, f_β, as an application which to every set of models $\mathcal{M}$ and every multi-set of sets of models $\mathcal{X}$ associates a set of models $f_\beta(\mathcal{M}, \mathcal{X})$ such that:*

1. $Cl_\beta(f_\beta(\mathcal{M}, \mathcal{X})) = f_\beta(\mathcal{M}, \mathcal{X})$ *($f_\beta(\mathcal{M}, \mathcal{X})$ is closed under β)*
2. $f_\beta(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$
3. *if $\mathcal{M} = Cl_\beta(\mathcal{M})$, then $f_\beta(\mathcal{M}, \mathcal{X}) = \mathcal{M}$*
4. *If $\mathcal{M} \neq \emptyset$, then $f_\beta(\mathcal{M}, \mathcal{X}) \neq \emptyset$.*

The concept of mappings allows us to define a family of refined operators for fragments of classical logic that captures the examples given before.

Definition 8. *Let $\Delta : \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$ be a merging operator and $\mathcal{L}' \subseteq \mathcal{L}$ be a β-fragment of classical logic with $\beta \in \mathcal{B}$. For a β-mapping f_β we denote with $\Delta^{f_\beta} : \mathcal{E}_{\mathcal{L}'} \times \mathcal{L}' \to \mathcal{K}_{\mathcal{L}'}$ the operator for $\mathcal{L}'$ defined as $\mathrm{Mod}(\Delta_\mu^{f_\beta}(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}od(E))$. The class $[\Delta, \mathcal{L}']$ contains all operators Δ^{f_β} where f_β is a β-mapping and $\beta \in \mathcal{B}$ such that $\mathcal{L}'$ is a β-fragment.*

The next proposition is central in reflecting that the above class captures all refined operators we had in mind, cf. Definition 5.

Proposition 2. *Let $\Delta : \mathcal{E}_\mathcal{L} \times \mathcal{L} \to \mathcal{K}_\mathcal{L}$ be a basic merging operator and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment of classical logic. Then, $[\Delta, \mathcal{L}']$ is the set of all Δ-refinements for $\mathcal{L}'$.*

Proof. Let $\mathcal{L}'$ be a β-fragment for some $\beta \in \mathcal{B}$. Let $\Delta^\star \in [\Delta, \mathcal{L}']$. We show that $\Delta^\star$ is a Δ-refinement for $\mathcal{L}'$. Let $\mu \in \mathcal{L}'$ and $E \in \mathcal{E}_{\mathcal{L}'}$. Since $\Delta^\star \in [\Delta, \mathcal{L}']$ there exists a β-mapping f_β, such that $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}od(E))$. By Property 1 in Definition 7 $\Delta^\star_\mu(E)$ is indeed in $\mathcal{K}_{\mathcal{L}'}$. Consistency: If $\mathrm{Mod}(\Delta_\mu(E)) \neq \emptyset$ then $\mathrm{Mod}(\Delta^\star_\mu(E)) \neq \emptyset$ by Property 4 in Definition 7. Otherwise, by Property 2 in Definition 7, we get $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) = Cl_\beta(\emptyset) = \emptyset$. Equivalence for $\Delta^\star$ is clear by definition and since f_β is defined on sets of models. Containment: let $\phi \in T_{\mathcal{L}'}(\Delta_\mu(E))$, i.e., $\phi \in \mathcal{L}'$ and $\mathrm{Mod}(\Delta_\mu(E)) \subseteq \mathrm{Mod}(\phi)$. We have $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) \subseteq Cl_\beta(\mathrm{Mod}(\phi))$ by monotonicity of Cl_β. By Property 2 of Definition 7, $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E)))$. Since $\phi \in \mathcal{L}'$ we have $Cl_\beta(\mathrm{Mod}(\phi)) = \mathrm{Mod}(\phi)$. Thus, $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq \mathrm{Mod}(\phi)$, i.e., $\phi \in T_{\mathcal{L}'}(\Delta^\star_\mu(E))$. Invariance: In case $\Delta_\mu(E) \in \mathcal{K}_{\langle \mathcal{L}' \rangle}$, we have $Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))) = \mathrm{Mod}(\Delta_\mu(E))$ since $\mathcal{L}'$ is a β-fragment. By Property 3 in Definition 7, we have $\mathrm{Mod}(\Delta^\star_\mu(E)) = f_\beta(\mathrm{Mod}(\Delta_\mu(E)), \mathcal{M}od(E)) = \mathrm{Mod}(\Delta_\mu(E))$. Thus $T_{\mathcal{L}'}(\Delta^\star_\mu(E)) \subseteq T_{\mathcal{L}'}(\Delta_\mu(E))$ as required.

Let $\Delta^\star$ be a Δ-refinement for $\mathcal{L}'$. We show that $\Delta^\star \in [\Delta, \mathcal{L}']$. Let f be defined as follows for any set $\mathcal{M}$ of interpretations and $\mathcal{X}$ a multi-set of sets of interpretations: $f(\emptyset, \mathcal{X}) = \emptyset$. For $\mathcal{M} \neq \emptyset$, if $Cl_\beta(\mathcal{M}) = \mathcal{M}$ then $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$, otherwise if there exists a pair $(E, \mu) \in (\mathcal{E}_{\mathcal{L}'}, \mathcal{L}')$ such that $\mathcal{M}od(E) = \mathcal{X}$ and $\mathrm{Mod}(\Delta_\mu(E)) = \mathcal{M}$, then we define $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$. If there is no such (E, μ) then we arbitrarily define $f(\mathcal{M}, \mathcal{X})$ as the set consisting of a single model, say the minimal model of $\mathcal{M}$ in the lexicographic order. Note that since $\Delta^\star$ is a Δ-refinement for $\mathcal{L}'$, it satisfies the property of equivalence, thus the actual choice of the pair (E, μ) is not relevant, and hence f is well-defined. Thus the refined operator $\Delta^\star$ behaves like the operator Δ^f.

We show that such a mapping f is a β-mapping. We show that the four properties in Definition 7 hold for f. Property 1 is ensured since for every pair $(\mathcal{M}, \mathcal{X})$, $f(\mathcal{M}, \mathcal{X})$ is closed under β. Indeed, either $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$ if $\mathcal{M}$ is closed under β, or $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$ and since $\Delta^\star_\mu(E) \in \mathcal{K}_{\mathcal{L}'}$ its set of models is closed under β, or $f(\mathcal{M}, \mathcal{X})$ consists of a single interpretation, and thus is also closed under β. Let us show Property 2, i.e., $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$ for any pair $(\mathcal{M}, \mathcal{X})$. It is obvious when $\mathcal{M} = \emptyset$ (then $f(\mathcal{M}, \mathcal{X}) = \emptyset$), as well as when $f(\mathcal{M}, \mathcal{X})$ is a singleton and when $\mathcal{M}$ is closed and thus $f(\mathcal{M}, \mathcal{X}) = \mathcal{M}$. Otherwise $f(\mathcal{M}, \mathcal{X}) = \mathrm{Mod}(\Delta^\star_\mu(E))$ and since $\Delta^\star$ satisfies containment $\mathrm{Mod}(\Delta^\star_\mu(E)) \subseteq Cl_\beta(\mathrm{Mod}(\Delta_\mu(E))$. Therefore in any case we have $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\beta(\mathcal{M})$. Property 3 follows trivially from the definition of $f(\mathcal{M}, \mathcal{X})$ when $\mathcal{M}$ is closed under β. Property 4 is ensured by consistency of $\Delta^\star$. □

Note that the β-mapping which is used in the characterization of refined merging operators differs from the one used in the context of revision (see [6]). Indeed, our mapping has two arguments (and not only one as in the case of revision). The additional multi-set of sets of models representing the profile is required to capture approaches like the Min/Cl_β-based refined operator, which are profile dependent.

4 IC Postulates

The aim of this section is to study whether refinements of merging operators preserve the IC postulates. We first show that in case the initial operator satisfies the most basic postulates ((IC0)–(IC3)), then so does any of its refinements. It turns out that this result can not be extended to the remaining postulates. For (IC4) we characterize a subclass of refinements for which this postulate is preserved. For the four remaining postulates we study two representative kinds of distance-based merging operators. We show that postulates (IC5) and (IC7) are violated for all of our proposed examples of refined operators with the exception of the Min-based refinement. For (IC6) and (IC8) the situation is even worse in the sense that no refinement of our proposed examples of merging operators can satisfy them neither for $\mathcal{L}_{Horn}$ nor for $\mathcal{L}_{Krom}$. Table 1 gives an overview of the results of this section. However, note that some of the forthcoming results are more general and hold for arbitrary fragments and/or operators.

Proposition 3. *Let Δ be a merging operator satisfying postulates* (IC0)–(IC3)*, and $\mathcal{L}' \subseteq \mathcal{L}$ be a characterizable fragment. Then each Δ-refinement for $\mathcal{L}'$ satisfies* (IC0)–(IC3) *in $\mathcal{L}'$ as well.*

A natural question is whether refined operators for characterizable fragments in their full generality preserve other postulates, and if not whether one can nevertheless find some refined operators that satisfy some of the remaining postulates.

First we show that one can not expect to extend Proposition 3 to (IC4). Indeed, in the two following propositions we exhibit merging operators which satisfy all postulates, whereas some of their refinements violate (IC4) in some fragments.

Proposition 4. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then the* Min*-based refined operator Δ^{Min} violates postulate* (IC4) *in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$. In case d is the drastic distance, Δ^{Min} violates postulate* (IC4) *in every characterizable fragment $\mathcal{L}' \subset \mathcal{L}$.*

Proposition 5. *Let $\Delta = \Delta^{d,\mathrm{GMax}}$ be a merging operator where d is an arbitrary non-drastic counting distance. Then the closure-based refined operator Δ^{Cl_β} violates* (IC4) *in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.*

In order to identify a class of refinements which satisfy (IC4), we now introduce the notion of fairness for Δ-refinements.

	$(\Delta^{d_H,\Sigma})^{Cl_\beta}$	$(\Delta^{d_H,\mathrm{GMax}})^{Cl_\beta}$	$(\Delta^{d_D,x})^{Cl_\beta}$	$(\Delta^{d,x})^{\mathrm{Min}}$	$(\Delta^{d,x})^{\mathrm{Min}/Cl_\beta}$
IC0 - IC3	+	+	+	+	+
IC4	+	-	+	-	+
IC5, IC7	-	-	-	+	-
IC6, IC8	-	-	-	-	-

Table 1. Overview of some results for (IC4)–(IC8) for refinements in the Horn and Krom fragment ($x \in \{\Sigma, \mathrm{GMax}\}$, $d \in \{d_H, d_D\}$).

Definition 9. *Let $\mathcal{L}'$ be a fragment of classical logic. A Δ-refinement for $\mathcal{L}'$, $\Delta^\star$, is* fair *if it satisfies the following property for each $E \in \mathcal{E}_{\mathcal{L}'}$, $\mu \in \mathcal{L}'$: If $\#(\Delta_\mu(E), E) \neq 1$ then $\#(\Delta^\star_\mu(E), E) \neq 1$.*

Proposition 6. *Let $\mathcal{L}'$ be a characterizable fragment. (1) The Cl_β-based refinement of both $\Delta^{d_D,\Sigma}$ and $\Delta^{d_D,\mathrm{GMax}}$ for $\mathcal{L}'$ is fair. (2) the Min/Cl_β-based refinement of any merging operator for $\mathcal{L}'$ is fair.*

Fairness turns out to be a sufficient property to preserve the postulate (IC4) as stated in the following proposition.

Proposition 7. *Let Δ be a merging operator satisfying postulate* (IC4)*, and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then every fair Δ-refinement for $\mathcal{L}'$ satisfies* (IC4) *as well.*

With the above result at hand, we can conclude that the Cl_β-based refinement of both $\Delta^{d_D,\Sigma}$ and $\Delta^{d_D,\mathrm{GMax}}$ for $\mathcal{L}'$ as well as the Min/Cl_β-based refinement of any merging operator satisfies (IC4).

Remark 1. *Observe that the distance which is used in distance-based operators matters with respect to the preservation of* (IC4)*, as well as for fairness. Indeed, while the Cl_β-refinement of $\Delta^{d_D,\mathrm{GMax}}$ is fair, and therefore satisfies* (IC4)*, the Cl_β-refinement of $\Delta^{d,\mathrm{GMax}}$ where d is an arbitrary non-drastic counting distance violates postulate* (IC4) *in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$, and therefore is not fair.*

For all refinements considered so far we know whether (IC4) is preserved or not, with one single exception: the Cl_β-refinement of $\Delta^{d,\Sigma}$ where d is an arbitrary non-drastic counting distance. In this case we get a partial positive result.

Proposition 8. *Let Δ be a merging operator with $\Delta = \Delta^{d,\Sigma}$, where d is an arbitrary counting distance that satisfies the triangular inequality. Then the closure-based refined operator Δ^{Cl_β} satisfies postulate* (IC4) *in any characterizable fragment.*

Remark 2. *The above proposition together with Proposition 5 shows that the aggregation function that is used in distance-based operators matters with respect to the preservation of the postulate* (IC4).

Interestingly Proposition 8 (recall that the Hamming distance satisfies the triangular inequality) together with the following proposition show that fairness, which is a sufficient condition for preserving (IC4) is not a necessary one.

Proposition 9. *The Cl_β-refinement of $\Delta^{d_H,\Sigma}$ is not fair in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.*

It turns out that our refined operators have a similar behavior with respect to postulates (IC5) & (IC7) as well as (IC6) & (IC8). Therefore we will deal with the remaining postulates in pairs. In fact the Min-based refinement satisfies (IC5) and (IC7), whereas the refined operators Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ violate these two postulates.

Proposition 10. *Let Δ be a merging operator satisfying postulates* (IC5) *and* (IC6) *(resp.* (IC7) *and* (IC8)*), and $\mathcal{L}' \subseteq \mathcal{L}$ a characterizable fragment. Then the refined operator Δ^{Min} for $\mathcal{L}'$ satisfies* (IC5) *(resp.* (IC7)*) in $\mathcal{L}'$ as well.*

Proposition 11. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then the refined operators Δ^{Cl_β} and $\Delta^{\mathrm{Min}/Cl_\beta}$ violate postulates* (IC5) *and* (IC7) *in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.*

Proof. We give the proof for Δ^{Cl_β} with $\Delta = \Delta^{d,\Sigma}$ where d is associated with a function g (see Definition 3). The given examples also apply to GMax and for the refinement $\Delta^{\mathrm{Min}/Cl_\beta}$.

(IC5): Let $\beta \in \{\wedge, \mathrm{maj}_3\}$. Consider $E_1 = \{K_1, K_2, K_3\}$, $E_2 = \{K_4\}$ and μ with $\mathrm{Mod}(K_1) = \{\{a\}, \{a,b\}, \{a,c\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}, \{a,b\}, \{b,c\}\}$, $\mathrm{Mod}(K_3) = \{\{c\}, \{a,c\}, \{b,c\}\}$, $\mathrm{Mod}(K_4) = \{\emptyset, \{b\}\}$, and $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$.

	K_1	K_2	K_3	K_4	E_1	$E_1 \sqcup E_2$
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	0	$3g(1)$	$3g(1)$
$\{a\}$	0	$g(1)$	$g(1)$	$g(1)$	$2g(1)$	$3g(1)$
$\{b\}$	$g(1)$	0	$g(1)$	0	$2g(1)$	$2g(1)$
$\{c\}$	$g(1)$	$g(1)$	0	$g(1)$	$2g(1)$	$3g(1)$

Since $g(1) > 0$ by definition of a counting distance, we have $\mathrm{Mod}(\Delta^{Cl_\beta}_\mu(E_1)) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, $\mathrm{Mod}(\Delta^{Cl_\beta}_\mu(E_2)) = \{\emptyset, \{b\}\}$, and $\mathrm{Mod}(\Delta^{Cl_\beta}_\mu(E_1 \sqcup E_2)) = \{\{b\}\}$, violating (IC5).

(IC7): For $\mathcal{L}_{Horn}$, consider $E = \{K_1, K_2, K_3\}$ with $\mathrm{Mod}(K_1) = \{\{a\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}\}$, $\mathrm{Mod}(K_3) = \{\{a,b\}\}$, and assume $\mathrm{Mod}(\mu_1) = \{\emptyset, \{a\}, \{b\}\}$ and $\mathrm{Mod}(\mu_2) = \{\emptyset, \{a\}\}$.

	K_1	K_2	K_3	E
$\emptyset$	$g(1)$	$g(1)$	$g(2)$	$2g(1)+g(2)$
$\{a\}$	0	$g(2)$	$g(1)$	$g(1)+g(2)$
$\{b\}$	$g(2)$	0	$g(1)$	$g(1)+g(2)$

We have $\mathrm{Mod}(\Delta_{\mu_1}(E)) = \{\{a\}, \{b\}\}$, thus $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1}(E)) = \{\emptyset, \{a\}, \{b\}\}$. Therefore, $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1}(E) \wedge \mu_2) = \{\emptyset, \{a\}\}$, whereas $\mathrm{Mod}(\Delta^{Cl_\wedge}_{\mu_1 \wedge \mu_2}(E)) = \{\{a\}\}$, violating (IC7).

For $\mathcal{L}_{Krom}$ let $E = \{K_1, K_2, K_3, K_4, K_5\}$, μ_1 and μ_2 with $\mathrm{Mod}(K_1) = \{\{a\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}\}$, $\mathrm{Mod}(K_3) = \{\{c\}\}$, $\mathrm{Mod}(K_4) = \{\{a,b\}, \{a,c\}\}$, $\mathrm{Mod}(K_5) = \{\{a,b\}, \{b,c\}\}$, $\mathrm{Mod}(\mu_1) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, and $\mathrm{Mod}(\mu_2) = \{\emptyset, \{a\}\}$.

	K_1	K_2	K_3	K_4	K_5	E
$\emptyset$	$g(1)$	$g(1)$	$g(1)$	$g(2)$	$g(2)$	$2g(2)+3g(1)$
$\{a\}$	0	$g(2)$	$g(2)$	$g(1)$	$g(1)$	$2g(2)+2g(1)$
$\{b\}$	$g(2)$	0	$g(2)$	$g(1)$	$g(1)$	$2g(2)+2g(1)$
$\{c\}$	$g(2)$	$g(2)$	0	$g(1)$	$g(1)$	$2g(2)+2g(1)$

We have $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1}(E)) = \{\emptyset, \{a\}, \{b\}, \{c\}\}$, thus $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1}(E) \wedge \mu_2) = \{\emptyset, \{a\}\}$, and $\mathrm{Mod}(\Delta^{Cl_{\mathrm{maj}_3}}_{\mu_1 \wedge \mu_2}(E)) = \{\{a\}\}$. This violates postulate (IC7). $\square$

Actually in the Horn fragment the negative results of the above proposition can be extended to any fair refinement.

Proposition 12. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then any fair refined operator Δ^* violates* (IC5) *and* (IC7) *in $\mathcal{L}_{Horn}$.*

We leave it as an open question whether this proposition can be extended to Krom. For the two remaining postulates, (IC6) and (IC8), the situation is even worse, since any refinement of the two kinds of operators we considered violates them in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.

Proposition 13. *Let Δ be a merging operator with $\Delta \in \{\Delta^{d,\Sigma}, \Delta^{d,\mathrm{GMax}}\}$, where d is an arbitrary counting distance. Then any refined operator $\Delta^\star$ violates postulates* (IC6) *and* (IC8) *in $\mathcal{L}_{Horn}$ and in $\mathcal{L}_{Krom}$.*

Proof. As an example we give the proof for (IC6) in $\mathcal{L}_{Horn}$ for $\Delta^{d,\mathrm{GMax}}$. Since $\mathcal{L}_{Horn}$ is an $\wedge$-fragment, there is an $\wedge$-mapping f such that $\Delta^\star = \Delta^f$ and we have $f(\mathcal{M}, \mathcal{X}) \subseteq Cl_\wedge(\mathcal{M})$ with $Cl_\wedge(f(\mathcal{M}, \mathcal{X})) = f(\mathcal{M}, \mathcal{X})$. Let us consider $E_1 = \{K_1, K_2, K_3\}$ and μ with $\mathrm{Mod}(K_1) = \{\{a\}, \{a, b\}\}$, $\mathrm{Mod}(K_2) = \{\{b\}, \{a, b\}\}$, $\mathrm{Mod}(K_3) = \{\emptyset, \{a\}, \{b\}\}$ and $\mathrm{Mod}(\mu) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$.

	K_1	K_2	K_3	E_1
$\emptyset$	$g(1)$	$g(1)$	0	$(g(1), g(1), 0)$
$\{a\}$	0	$g(1)$	0	$(g(1), 0, 0)$
$\{b\}$	$g(1)$	0	0	$(g(1), 0, 0)$
$\{a, b\}$	0	0	$g(1)$	$(g(1), 0, 0)$

We have $\mathcal{M} = \mathrm{Mod}(\Delta_\mu(E_1)) = \{\{a\}, \{b\}, \{a, b\}\}$. Let us consider the possibilities for $\mathrm{Mod}(\Delta^\star_\mu(E_1)) = f(\mathcal{M}, \mathcal{M}od(E_1))$. If $\emptyset \in f(\mathcal{M}, \mathcal{M}od(E_1))$, then let $E_2 = \{K_4\}$ with K_4 in $\mathcal{L}_{Horn}$ be such that $\mathrm{Mod}(K_4) = \{\emptyset\}$. Thus, $\mathrm{Mod}(\Delta^\star_\mu(E_2)) = \{\emptyset\}$ and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\emptyset\}$. Moreover, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\emptyset, \{a\}, \{b\}\}$ or $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ depending on whether $g(1) < g(2)$ or $g(1) = g(2)$. Since both sets are closed under intersection, we have $\mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2)) = \mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2))$. Thus $\mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2)) \not\subseteq \{\emptyset\}$ and (IC6) does not hold.

Otherwise, $f(\mathcal{M}, \mathcal{M}od(E_1)) \subseteq \{\{a\}, \{b\}, \{a, b\}\}$. By symmetry assume w.l.o.g. that $f(\mathcal{M}, \mathcal{M}od(E_1)) \subseteq \{\{a, b\}, \{a\}\}$ (note that $\{\{a\}, \{b\}\} \subseteq f(\mathcal{M}, \mathcal{M}od(E_1))$ would imply $\emptyset \in f(\mathcal{M}, \mathcal{M}od(E_1))$). If $f(\mathcal{M}, \mathcal{M}od(E_1)) = \{\{a\}\}$ or $\{\{a, b\}\}$, then let $E_2 = \{K_1\}$. Then, $\mathrm{Mod}(\Delta_\mu(E_2)) = \{\{a\}, \{a, b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_2))$, and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\{a\}\}$ or $\{\{a, b\}\}$. Furthermore, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\{a\}, \{a, b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2))$, thus violating (IC6). If $f(\mathcal{M}, \mathcal{M}od(E_1)) = \{\{a, b\}, \{a\}\}$, then let $E_2 = \{K_2\}$. Then, $\mathrm{Mod}(\Delta_\mu(E_2)) = \{\{b\}, \{a, b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_2))$, and $\mathrm{Mod}(\Delta^\star_\mu(E_1) \wedge \Delta^\star_\mu(E_2)) = \{\{a, b\}\}$. Furthermore, $\mathrm{Mod}(\Delta_\mu(E_1 \sqcup E_2)) = \{\{b\}, \{a, b\}\} = \mathrm{Mod}(\Delta^\star_\mu(E_1 \sqcup E_2))$, and thus (IC6) does not hold. □

5 Conclusion

We have investigated to which extent known merging operators can be refined to work within fragments of propositional logic. Compared to revision, this task is more involved since merging operators have many parameters that have to be taken into account.

We have first defined desired properties any refined merging operator should satisfy and provided a characterization of all refined merging operators. We have shown that the refined merging operators preserve the basic postulates, namely (IC0)–(IC3). The situation is more complex for the other postulates. For the postulate (IC4) we have provided a sufficient condition for its preservation by a refinement (fairness). For the other postulates, we have focused on two representative families of distance-based merging operators that satisfy the postulates (IC0)–(IC8). For these two families the preservation of (IC5) and (IC7) depends on the used refinement and it would be interesting to obtain a necessary and sufficient condition for this. In contrast, there is no hope for such a condition for (IC6) and (IC8), since we have shown that any refinement of merging operators belonging to these families violates these postulates in $\mathcal{L}_{Horn}$ and $\mathcal{L}_{Krom}$.

An interesting issue is whether the postulate (IC4) is compatible with (IC5) and (IC7) for some refinements and whether this can depend on the fragment under consideration.

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REFERENCES

[1] C.E. Alchourrón, P. Gärdenfors, and D. Makinson, 'On the logic of theory change: Partial meet contraction and revision functions', *J. Symb. Log.*, **50**(2), 510–530, (1985).
[2] C. Baral, S. Kraus, and J. Minker, 'Combining multiple knowledge bases', *IEEE Trans. Knowl. Data Eng.*, **3**(2), 208–220, (1991).
[3] I. Bloch and A. Hunter (Eds), 'Fusion: General concepts and characteristics', *Int. J. Intell. Syst.*, **16**(10), 1107–1134, (2001).
[4] R. Booth, T.A. Meyer, I.J. Varzinczak, and R. Wassermann, 'On the link between partial meet, kernel, and infra contraction and its application to Horn logic', *J. Artif. Intell. Res.*, **42**, 31–53, (2011).
[5] L. Cholvy, 'Reasoning about merging information', *Handbook of DRUMS*, **3**, 233–263, (1998).
[6] N. Creignou, O. Papini, R. Pichler, and S. Woltran, 'Belief revision within fragments of propositional logic', *J. Comput. Syst. Sci.*, **80**(2), 427–449, (2014). (Preliminary version in Proc. KR, 2012).
[7] J.P. Delgrande and P. Peppas, 'Revising Horn theories', in *Proc. IJCAI*, pp. 839–844, (2011).
[8] J.P. Delgrande and R. Wassermann, 'Horn clause contraction functions', *J. Artif. Intell. Res.*, **48**, 475–511, (2013).
[9] S. Konieczny, J. Lang, and P. Marquis, 'DA^2 merging operators', *Artif. Intell.*, **157**(1-2), 49–79, (2004).
[10] S. Konieczny and R. Pino Pérez, 'Merging information under constraints: A logical framework', *J. Log. Comput.*, **12**(5), 773–808, (2002).
[11] S. Konieczny and R. Pino Pérez, 'Logic based merging', *J. Philosophical Logic*, **40**(2), 239–270, (2011).
[12] J. Lin, 'Integration of weighted knowledge bases', *Artif. Intell.*, **83**(2), 363–378, (1996).
[13] J. Lin and A.O. Mendelzon, 'Merging databases under constraints', *Int. J. Cooperative Inf. Syst.*, **7**(1), 55–76, (1998).
[14] F. Van De Putte, 'Prime implicates and relevant belief revision', *J. Log. Comput.*, **23**(1), 109–119, (2013).
[15] P.Z. Revesz, 'On the semantics of theory change: Arbitration between old and new information', in *Proc. PODS*, pp. 71–82, (1993).
[16] P.Z. Revesz, 'On the semantics of arbitration', *IJAC*, **7**(2), 133–160, (1997).
[17] T.J. Schaefer, 'The complexity of satisfiability problems', in *Proc. STOC*, pp. 216–226, (1978).
[18] Z.Q. Zhuang and M. Pagnucco, 'Model based Horn contraction', in *Proc. KR*, pp. 169–178, (2012).
[19] Z.Q. Zhuang, M. Pagnucco, and Y. Zhang, 'Definability of Horn revision from Horn contraction', in *Proc. IJCAI*, (2013).