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Convex Solutions of RCC8 Networks

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Abstract. RCC8 is one of the most widely used calculi for qualitative spatial reasoning. Although many applications have been explored where RCC8 relations refer to geographical or physical regions in two- or three-dimensional spaces, their use for conceptual reasoning is still at a rather preliminary stage. One of the core obstacles with using RCC8 to reason about conceptual spaces is that regions are required to be convex in this context. We investigate in this paper how the latter requirement impacts the realizability of RCC8 networks. Specifically, we show that consistent RCC8 networks over $2n + 1$ variables are guaranteed to have a convex solution in Euclidean spaces of n dimensions and higher. We furthermore prove that our bound is optimal for 2- and 3-dimensional spaces, and that for any number of dimensions $n \geq 4$, there exists a network of RCC8 relations over $3n$ variables which is consistent, but does not allow a convex solution in the n-dimensional Euclidean space.

1 Introduction

RCC8 is a constraint language for expressing qualitative mereo-topological relations between regions. It originates from the influential Region Connection Calculus (RCC), which was introduced in [9] as a first-order theory for defining mereo-topological relations, starting from the notion of connection as the only primitive spatial relation. In RCC8, eight jointly exhaustive and pairwise disjoint base relations are used to express the spatial relationship of two regions

$$\mathcal{R}_8 = \{\mathbf{EQ}, \mathbf{DC}, \mathbf{EC}, \mathbf{PO}, \mathbf{TPP}, \mathbf{NTPP}, \mathbf{TPP}^{-1}, \mathbf{NTPP}^{-1}\}$$

The intended meaning of six of these relations is illustrated in Table 1, with the remaining relations being defined by $a\, \mathbf{NTPP}^{-1}\, b$ iff $b\, \mathbf{NTPP}\, a$, and $a\, \mathbf{TPP}^{-1}\, b$ iff $b\, \mathbf{TPP}\, a$. An RCC8 network $\Theta = \{v_i\, \sigma_{ij}\, v_j \mid (v_i, v_j) \in V^2\}$ over a set of variables V defines for each pair of variables $(v_i, v_j) \in V^2$ a subset σ_{ij} of relations from $\mathcal{R}_8$, containing those relations that are allowed to hold between u and v. A basic RCC8 network is an RCC8 network in which each of these subsets $\sigma_{ij} = \{R_{ij}\}$ is a singleton. Slightly abusing notation, we write such a network as $\Theta = \{v_i\, R_{ij}\, v_j \mid (v_i, v_j) \in V^2\}$. For the ease of presentation, we also write $\mathbf{PP}$ for $\{\mathbf{TPP}, \mathbf{NTPP}\}$ and $\mathbf{DR}$ for $\{\mathbf{EC}, \mathbf{DC}\}$. A network Θ over $V = \{v_1, ..., v_n\}$ is called consistent if there exists a mapping from the variables v_i to regular closed [3] regions X_i, taken from a topological space that satisfies the axioms of the RCC [9], such that the RCC8 relation which holds between X_i and X_j is among those in σ_{ij} for all $(v_i, v_j) \in V^2$. Such a mapping is called a solution. A solution $\mathcal{S}$ is said to be k-dimensional (resp. convex) if each of the X_i is a k-dimensional region in $\mathbb{R}^k$ (resp. a convex region). Deciding whether an RCC8 network is consistent is NP-complete in general, although this problem becomes polynomial in the case of basic networks [11].

Table 1. Six of the eight possible RCC8 relations that can hold between two regions a and b with interiors $i(a)$ and $i(b)$.

Name	Symbol	Meaning
Equals	**EQ**	$a = b$
Disconnected	**DC**	$a \cap b = \emptyset$
Externally Connected	**EC**	$a \cap b \neq \emptyset$, $i(a) \cap i(b) = \emptyset$
Partially Overlap	**PO**	$a \cap b \neq \emptyset$, $a \not\subseteq b$, $b \not\subseteq a$
Tangential Proper Part	**TPP**	$a \subset b$, $a \not\subset i(b)$
Non-Tangential Proper Part	**NTPP**	$a \subset i(b)$

A remarkable result about RCC8 is that any consistent network can be realized in Euclidean spaces of arbitrary dimension $n \geq 1$ [10], and even in the digital plane $\mathbb{Z}^2$ [8]. This means that neither the restriction to Euclidean, or even discrete spaces, nor the restriction to spaces of a particular dimension puts additional constraints on the realizability of RCC8 networks. This situation changes, however, when more demanding properties are required of regions. For example, when regions are required to be self-connected[4], some consistent RCC8 networks cannot be realized in $\mathbb{R}$ or $\mathbb{R}^2$, although all consistent RCC8 networks can be realized using self-connected regions in $\mathbb{R}^n$ for any $n \geq 3$ [10]. Deciding whether a basic RCC8 network can be realized by self-connected regions in $\mathbb{R}^2$ is an NP-complete problem [7, 13]. If we can specify for each region whether it has a connected interior, then [6] shows that the satisfaction problem becomes undecidable even over polygons in $\mathbb{R}^2$.

As convexity implies self-connectedness, it follows that some consistent RCC8 networks cannot be realized by convex regions in $\mathbb{R}^2$. Even worse, for any number of dimensions n, there exist consistent RCC8 networks that are not realizable in $\mathbb{R}^n$ using convex regions [3]. Deciding whether an RCC8 network can be realized by convex regions in $\mathbb{R}^n$ was shown in [3] to be as hard as checking the consistency of algebraic constraints over the real numbers. However, as we show in this paper, this only holds if the number of dimensions is fixed a priori: for any consistent RCC8 network Θ, there exists an n such that Θ is realizable by convex regions in $\mathbb{R}^n$. Specifically, we show that consistent RCC8 networks over up to $2n+1$ variables can always be realized by convex regions in $\mathbb{R}^n$ if $n \geq 2$. Note that the case where $n = 1$ is different, but straightforward: any consistent network over 2 variables is realizable by intervals in R, but not e.g. $\{v_1\, \mathbf{EC}\, v_2, v_2\, \mathbf{EC}\, v_3, v_1\, \mathbf{EC}\, v_3\}$.

Our work is motivated by applications which use RCC8 relations to reason about conceptual spaces [4]. Conceptual spaces are metric spaces in which the meaning of natural language properties can

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[3] A set X in a topological space with interior operator i and closure operator cl is said to be regular closed if $X = cl(i(X))$.

[4] A region is said be self-connected if between any two points of the region there is a continuous path which belongs to the region.

be expressed. Despite being a simplification, it is common to identify conceptual spaces with (typically high-dimensional) Euclidean spaces [5], where each of the dimensions corresponds to a cognitively primitive quality. In a conceptual space of colours, for example, there could be three dimensions, corresponding to hue, intensity and saturation. Central in the theory of conceptual spaces is the assumption that the representation of natural properties should always be a convex region, a view which is close in spirit to the assumption underlying prototype theory [12] that the membership of an object to a category depends on its distance to the prototypes of that category, relative to its distance to the prototypes of other categories. While the theory of conceptual spaces can be used to explain a variety of cognitive phenomena in an elegant way, its application to knowledge representation is hampered by the fact that conceptual space representations are usually not available; in fact, for many domains it is not even known what the relevant dimensions would be.

This means that in practice we are often left with only qualitative descriptions. For instance, without access to the geometric representation of the properties *orange* and *red*, we may still assert that we should have *red* $\mathbf{EC}$ *orange*. The relation $\mathbf{EC}$ here reflects the view that there are borderline cases of colours for which it is hard to determine whether they are red or orange, while we may insist that orange and red are disjoint colours. Of course, the exact boundaries of what it means for an object to be orange may be vague, but we may still use an RCC8 based representation in such a case [5], by relying on the ideas of the Egg-Yolk calculus [2]. As another example, we may insist that *italianRestaurant* $\mathbf{NTPP}$ *restaurant*, without knowing any details on how ‘restaurant’ or ‘Italian restaurant’ would be represented in a conceptual space; all we are claiming in such a case is that all Italian restaurants are restaurants, and borderline cases of Italian restaurants would still be typical restaurants (but e.g. serving food which borders on French cuisine). On the other hand, we may assert that *bistro* $\mathbf{TPP}$ *restaurant* to encode that there are borderline instances of ‘bistro’ that would also be borderline instances of ‘restaurant’ (e.g. bars which also serve some food). Such qualitative models of conceptual spaces representations may be all that is needed in a knowledge representation setting. For instance, [14] shows how knowledge about the qualitative relation between different properties can be used to refine standard approaches for merging conflicting knowledge bases.

The results we present in this paper reveal that standard methods for reasoning about RCC8 relations can still be used to reason about conceptual space representations, provided that the underlying conceptual space is high-dimensional. Although this is the case for most domains, we would still need ad hoc procedures for sound and complete reasoning about simpler domains, such as the domain of colours (when there are more than $2n + 1 = 7$ variables).

2 Convex solutions for $2n + 1$ variables

In this section, we present a constructive proof for the following result:

Theorem 1. *Let Θ be a consistent RCC8 network over the set of variables V. If $|V| = 2n + 1$, with $n \geq 2$, it holds that Θ can be realized by convex regions in $\mathbb{R}^n$.*

Without lack of generality, we restrict our attention to basic networks, and we write R_{ij} for the relation that is imposed by Θ between v_i and v_j. We first show in Section 2.1 that Θ can be realized in the plane if $R_{ij} \in \{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$ for $i \neq j$. We subsequently use this result in Section 2.2 to show that Θ can be realized in $\mathbb{R}^n$ when $R_{ij} \in \{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$ for $i \neq j$. Finally, we consider arbitrary basic networks in Section 2.3.

2.1 Networks over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$

Suppose $\Theta = \{v_i\, R_{ij}\, v_j \mid (v_i, v_j) \in V^2\}$ is a consistent RCC8 network with $R_{ij} \in \{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$. We show that Θ is realizable in the plane using convex regions, regardless of the number of regions $|V| = k$. Without lack of generality, we can assume that $R_{ij} = \mathbf{TPP}$ only for $i < j$. Choose $k + 1$ different points $P_0, P_1, ..., P_k$, in that order, along the boundary of the unit circle, in the first quadrant. We write

$$a_i = \triangle OP_0P_i \tag{1}$$

$$b_i = CH(a_i \cup \bigcup \{a_j \mid (x_j\, \mathbf{TPP}\, x_i)\}) \tag{2}$$

where $\triangle ABC$ is the triangle defined by the points A, B and C, and $CH(X)$ is the convex hull of X. It is easy to see that b_i is a convex region which has the following property:

$$P_j \in b_i \text{ if and only if } j = 0, j = i, \text{ or } x_j\, \mathbf{TPP}\, x_i. \tag{3}$$

Now it is straightforward to show that $b_1, ..., b_k$ is a solution of Θ.

2.2 Networks over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$

Suppose $\Theta = \{v_i\, R_{ij}\, v_j \mid (v_i, v_j) \in V^2\}$ is a consistent RCC8 network with $R_{ij} \in \{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$. We write $\Theta{\downarrow}V' = \{v_i\, R_{ij}\, v_j \mid (v_i\, R_{ij}\, v_j) \in \Theta, (v_i, v_j) \in V'^2\}$ for the network obtained by restricting Θ to the variables from V' (with $V' \subseteq V$). Let us choose V' such that $V' \cup \{v_{i_1}, ..., v_{i_k}\} \cup \{v_{j_1}, ..., v_{j_k}\} \cup \{v_\omega\}$ constitutes a partition of V such that

1. $\Theta{\downarrow}V'$ is a network over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$;
2. $R_{i_l j_l} = \mathbf{EC}$ for each $l \in \{1, ..., k\}$;
3. $R_{i\omega} \neq \mathbf{TPP}$, i.e. no region is contained in v_ω.

It is clear that such a partition always exists; it suffices to first select a suitable v_ω and then repeatedly remove pairs of variables (v_{i_l}, v_{j_l}) from V for which $R_{i_l j_l} = \mathbf{EC}$.

For our purposes, we also need a weak notion of realization. We say that $R_{ij} = \mathbf{EC}$ is *weakly satisfied* by a mapping $\mathcal{S} = \{v_1^{X_1}, ..., v_n^{X_n}\}$ if $X_i\, \mathbf{DR}\, X_j$ holds, $R_{ij} = \mathbf{TPP}$ is weakly satisfied if $X_i\, \mathbf{PP}\, X_j$ holds and $R_{ij} = \mathbf{PO}$ is weakly satisfied if $X_i\, \mathbf{PO}\, X_j$ holds. We say that $\mathcal{S}$ is a *weak solution* of Θ if it weakly satisfies R_{ij} for each $(v_i, v_j) \in V^2$.

From Section 2.1, we know that $\Theta{\downarrow}V'$ can be realized in the plane. Starting from that two-dimensional representation, we first show in Section 2.2.1 that a $(k+2)$-dimensional convex weak solution exists for $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_k}, v_{j_1}, ..., v_{j_k}\})$. In Section 2.2.2, we use this $(k + 2)$-dimensional weak solution to define a $(k + 3)$-dimensional convex solution of Θ. If $|V'| \geq 6$, we have $k = \frac{2n-|V'|}{2} \leq n - 3$ and we have shown that Θ is realizable by convex regions in $\mathbb{R}^n$. The case where $|V'| \in \{0, 2, 4\}$ is treated separately in Section 2.2.3.

2.2.1 *Adding the regions $v_{i_{l+1}}$ and $v_{j_{l+1}}$*

Assume that we have an $(l + 2)$-dimensional convex weak solution $\mathcal{S}$ of $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_l}, v_{j_1}, ..., v_{j_l}\})$ with $0 \leq l \leq k - 1$. Let us write $V^* = V' \cup \{v_{i_1}, ..., v_{i_l}, v_{j_1}, ..., v_{j_l}\}$. We now construct an $(l + 3)$-dimensional convex weak solution $\mathcal{S}'$ of $\Theta{\downarrow}(V^* \cup$

$\{v_{i_{l+1}}, v_{j_{l+1}}\})$. Let us define the convex regions L and M in $\mathbb{R}^{l+2}$ as follows

$$L = CH\Big(\bigcup_{v_s \in V^*} \{\mathcal{S}(v_s)\}\Big) \cap \bigcap_{v_s \in V^*} \{\mathcal{S}(v_s) | R_{i_{l+1}s} = \mathbf{TPP}\}$$

$$M = CH\Big(\bigcup_{v_s \in V^*} \{\mathcal{S}(v_s)\}\Big) \cap \bigcap_{v_s \in V^*} \{\mathcal{S}(v_s) | R_{j_{l+1}s} = \mathbf{TPP}\}$$

Note that $L \neq \emptyset$ and $M \neq \emptyset$ since $|V^*| \geq |V'| \geq 6$ by assumption. We define

$$\mathcal{S}'(v_{i_{l+1}}) = \{(p_1, ..., p_{l+2}, \lambda) \,|\, (p_1, ..., p_{l+2}) \in L, \lambda \in [a, b]\}$$
$$\mathcal{S}'(v_{j_{l+1}}) = \{(p_1, ..., p_{l+2}, \lambda) \,|\, (p_1, ..., p_{l+2}) \in M, \lambda \in [-b, -a]\}$$

with $0 < a < b$ arbitrary but fixed values. For $v_s \in V^*$ we define

$$\mathcal{S}'(v_s) = \{(p_1, ..., p_{l+2}, \lambda) \,|\, (p_1, ..., p_{l+2}) \in \mathcal{S}(v_s), \lambda \in [v_s^-, v_s^+]\}$$

where

$$v_s^+ = \begin{cases} e & \text{If } R_{si_{l+1}} = \mathbf{TPP} \\ -a & \text{If } R_{i_{l+1}s} = \mathbf{EC} \text{ and } R_{sj_{l+1}} = \mathbf{TPP} \\ a & \text{If } R_{i_{l+1}s} = \mathbf{EC} \text{ and } R_{sj_{l+1}} \neq \mathbf{TPP} \\ e & \text{If } R_{i_{l+1}s} = \mathbf{PO} \\ b & \text{If } R_{i_{l+1}s} = \mathbf{TPP} \end{cases} \tag{4}$$

$$v_s^- = \begin{cases} -e & \text{If } R_{sj_{l+1}} = \mathbf{TPP} \\ a & \text{If } R_{j_{l+1}s} = \mathbf{EC} \text{ and } R_{si_{l+1}} = \mathbf{TPP} \\ -a & \text{If } R_{j_{l+1}s} = \mathbf{EC} \text{ and } R_{si_{l+1}} \neq \mathbf{TPP} \\ -e & \text{If } R_{j_{l+1}s} = \mathbf{PO} \\ -b & \text{If } R_{j_{l+1}s} = \mathbf{TPP} \end{cases} \tag{5}$$

with $a < e < b$.

Lemma 1. *The mapping $\mathcal{S}'$ is an $(l+3)$-dimensional convex weak solution of $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_{l+1}}, v_{j_1}, ..., v_{j_{l+1}}\})$.*

Proof. It is clear, by construction, that the relations $R_{i_{l+1}s}$ and $R_{j_{l+1}s}$ are weakly satisfied by $\mathcal{S}'$ for all $v_s \in V^*$, and moreover that $R_{i_{l+1}j_{l+1}}$ is weakly satisfied. We now show that also R_{st} is weakly satisfied for all v_s and v_t in V^* (with $s \neq t$).

It is clear that the relation $\mathbf{DR}$ is preserved under the proposed "cylindrical" extension of $\mathcal{S}$ to $\mathcal{S}'$.

If $\mathcal{S}(v_s)\,\mathbf{PO}\,\mathcal{S}(v_t)$ holds, we will have $\mathcal{S}'(v_s)\,\mathbf{PO}\,\mathcal{S}'(v_t)$ as soon as $[v_s^-, v_s^+] \cap [v_t^-, v_t^+]$ is a nondegenerate interval. The construction is such that $[-a, a] \subseteq [v_s^-, v_s^+] \cap [v_t^-, v_t^+]$, unless one of $R_{si_{l+1}}$, $R_{sj_{l+1}}$, $R_{ti_{l+1}}$ or $R_{tj_{l+1}}$ is $\mathbf{TPP}$. Assume for instance $R_{si_{l+1}} = \mathbf{TPP}$; the other cases are entirely analogous. Then we have $[v_s^-, v_s^+] = [a, e]$ and $R_{i_{l+1}t} = \mathbf{PO}$ and hence $[a, e] \subseteq [v_t^-, v_t^+]$, which means $[v_s^-, v_s^+] \cap [v_t^-, v_t^+] = [a, e]$.

If $\mathcal{S}(v_s)\,\mathbf{PP}\,\mathcal{S}(v_t)$ holds, we have $\mathcal{S}'(v_s)\,\mathbf{PP}\,\mathcal{S}'(v_t)$ as soon as $v_s^- \geq v_t^-$ and $v_s^+ \leq v_t^+$. We show that $v_s^+ \leq v_t^+$; the proof for $v_s^- \geq v_t^-$ is entirely analogous. If $R_{ti_{l+1}} = \mathbf{TPP}$, we will have $R_{si_{l+1}} = \mathbf{TPP}$, hence $v_s^+ = v_t^+ = e$. If $R_{tj_{l+1}} = \mathbf{TPP}$, we will have $R_{sj_{l+1}} = \mathbf{TPP}$, hence $v_s^+ = v_t^+ = -a$. If $R_{i_{l+1}t} = \mathbf{EC}$ and $R_{tj_{l+1}} \neq \mathbf{TPP}$, we will have $v_t^+ = a$ and $R_{si_{l+1}} = \mathbf{EC}$, which means $v_s^+ \in \{-a, a\}$ and thus $v_s^+ \leq v_t^+$. If $R_{i_{l+1}t} = \mathbf{PO}$, we will have $v_t^+ = e$, and $R_{i_{l+1}s} \neq \mathbf{TPP}$, which means $v_s^+ \leq e = v_t^+$. Finally, if $R_{i_{l+1}t} = \mathbf{TPP}$, then $v_t^+ = b$ and we trivially have $v_s^+ \leq v_t^+$.

□

2.2.2 Adding v_ω

After repeated applications of the procedure from Section 2.2.1, we end up with a convex $(k+2)$-dimensional weak solution $\mathcal{S}_k$ of $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_k}, v_{j_1}, ..., v_{j_k}\})$. If v_ω is contained in at least one other region from V, we can even obtain a $(k+2)$-dimensional weak solution of Θ in this way. Indeed, let e.g. $v_{i_{l+1}}$ be the region with smallest index l for which $R_{\omega i_{l+1}} = \mathbf{TPP}$. When adding the realization of $v_{i_{l+1}}$, we can add a representation of v_ω at the same time, as:

$$\mathcal{S}'(v_\omega) = \{(p_1, ..., p_{l+2}, \lambda) \,|\, (p_1, ..., p_{l+2}) \in L, \lambda \in [a', b]\}$$

for $a < a' < b$. The definition of (4)–(5) then needs to be adapted to differentiate e.g. between regions v_s that partial overlap both $v_{i_{l+1}}$ and v_ω from regions that only partially overlap $v_{i_{l+1}}$. We omit the details.

We now use this weak solution to construct a $(k+3)$-dimensional solution $\mathcal{S}^*$ of Θ. If there are no regions v_s in V for which $R_{\omega s} = \mathbf{TPP}$, we also realize v_ω at this step. In that case, we define the convex region $N \subseteq \mathbb{R}^{k+2}$ as

$$N = CH\Big(\bigcup_{v_s \in V_\omega} \{\mathcal{S}_k(v_s)\}\Big)$$

where $V_\omega = V \setminus \{v_\omega\}$, and $0 < a < e < b$ as before. We define

$$\mathcal{S}^*(v_\omega) = \{(\lambda \cdot p_1, ..., \lambda \cdot p_{k+2}, \lambda) \,|\, (p_1, ..., p_{k+2}) \in N, \lambda \in [a, b]\}$$

and for $v_s \in V_\omega$ we define $\mathcal{S}^*(v_s)$ as

$$\{(\lambda \cdot p_1, ..., \lambda \cdot p_{k+2}, \lambda) \,|\, (p_1, ..., p_{k+2}) \in \mathcal{S}_k(v_s), \lambda \in [0, v_s^*]\}$$

where $v_s^* = e$ if $R_{\omega s} = \mathbf{PO}$ and $v_s^* = a$ if $R_{\omega s} = \mathbf{EC}$. If V_ω was already realized in the $(k+2)$-dimensional weak solution $\mathcal{S}_k$, we simply define for each $v \in V$:

$$\{(\lambda \cdot p_1, ..., \lambda \cdot p_{k+2}, \lambda) \,|\, (p_1, ..., p_{k+2}) \in \mathcal{S}_k(v), \lambda \in [0, a]\}$$

Lemma 2. *The mapping $\mathcal{S}_\omega$ is a $(k+3)$-dimensional convex solution of Θ.*

Proof. This follows easily from the fact that all regions now meet at the origin, with the possible exception of v_ω, for which the required relations are clearly satisfied by construction. □

2.2.3 $|V'| \in \{0, 2, 4\}$

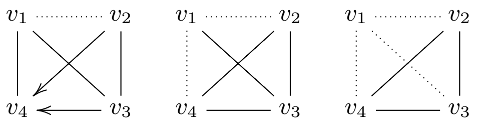
We now consider the special cases where V' has at most four variables. To this end, we need several lemmas. We first recall that a network of Interval Algebra (IA) [1] constraints is consistent if it has a solution using convex intervals.

When interpreted over convex intervals, RCC8 relations can be regard as disjunctions of basic IA relations. For example, **DC** is the disjunction of IA relations *before* and *after*. We say a basic IA constraint λ is a *refinement* of an RCC8 constraint θ if $\lambda \subseteq \theta$ when interpreted over convex intervals.

Lemma 3. *Let $\Theta = \{v_i\, R_{ij}\, v_j \,|\, (v_i, v_j) \in V^2\}$ be an RCC8 constraint network. Then Θ has a convex solution in $\mathbb{R}$ iff there exists a consistent basic IA constraint network $\Theta' = \{v_i\, \alpha_{ij}\, v_j \,|\, (v_i, v_j) \in V^2\}$ which is a refinement of Θ, i.e. $\alpha_{ij} \subseteq R_{ij}$ for all $i \neq j$.*

Therefore, to determine if an RCC8 constraint network has a realization using convex intervals, we need only determine if it has a consistent atomic IA refinement.

Lemma 4. *Let Θ be an RCC8 constraint network over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$ that involves four variables. Suppose Θ is consistent but has no convex weak solution in $\mathbb{R}$. Then Θ is isomorphic to one of the following three networks, where an arrow, a straight line, and a dotted line, represent, respectively, a* **TPP**, **PO**, *and* **EC** *relation.*



(N_4^1) (N_4^2) (N_4^3)

Proof. This can be checked by case-by-case analysis. We verified this using a computer program. □

We note that each N_4^i $(i = 1, 2, 3)$ contains an **EC** relation. The following result is clear.

Corollary 1. *Let Θ be an RCC8 constraint network over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$ that involves four variables. Suppose Θ is consistent. Then Θ has a convex weak solution in $\mathbb{R}$.*

Corollary 2. *Let Θ be an RCC8 constraint network over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$ that involves five variables. Suppose Θ is consistent. Then Θ has a solution using convex plane regions.*

Proof. Let $V = \{v_1, \cdots, v_5\}$ and write $V_{-i} = \{v_j \mid 1 \leq j \neq i \leq 5\}$. We prove that there exists an i such that v_i is minimal and $\Theta{\downarrow}V_{-i}$ is not isomorphic to any N_4^i $(i = 1, 2, 3)$. Without lack of generality, suppose v_5 is minimal but $\Theta{\downarrow}V_{-5}$ is isomorphic to e.g. N_4^3. By $R_{12} = \mathbf{EC}$, we know R_{51} and R_{52} cannot both be **TPP**. Therefore, either v_1 or v_2 is minimal. If v_1 is not minimal, then v_2 is minimal and $\Theta{\downarrow}V_{-2}$ is not isomorphic to N_4^i, and vice versa. If both are minimal, then if $R_{54} \neq \mathbf{PO}$, then $\Theta{\downarrow}V_{-1}$ is not isomorphic to any N_4^i; if $R_{54} = \mathbf{PO}$, then $\Theta{\downarrow}V_{-2}$ is not isomorphic to any N_4^i. The other cases are similar.

Now suppose v_i is minimal and $\Theta{\downarrow}V_{-i}$ is not isomorphic to any N_4^i. By Lemma 4, we know $\Theta{\downarrow}V_{-i}$ has a convex weak solution in $\mathbb{R}$. Using the method described in Section 2.2.2, we can extend the solution to v_i in two dimensions using convex regions. □

Now we return to the special cases when $|V'| \leq 4$. When $|V'| = 4$, by Corollary 1, we know $\Theta{\downarrow}V'$ has a convex weak solution in $\mathbb{R}$ since all relations are taken from $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}\}$. If $V' = \varnothing$ and $k \geq 2$, then $\Theta{\downarrow}\{a, b, c, d\}$ is not isomorphic to any of N_4^i $(i = 1, 2, 3)$, where $a, b, c, d \in V \setminus V'$ and $a\,\mathbf{EC}\,b$ and $c\,\mathbf{EC}\,d$. By Lemma 4, $\Theta{\downarrow}\{a, b, c, d\}$ has a convex weak solution in $\mathbb{R}$.

If $|V'| = 2$ and $k = 1$, then by Corollary 2 the network over $V' \cup \{v_{i_1}, v_{j_1}, v_\omega\}$ has a convex solution in $\mathbb{R}^2$.

Suppose $|V'| = 2$ and $k \geq 2$. We show the network over $\{a, b, c, d, e, f\}$ has a convex weak solution in $\mathbb{R}^2$, where $a, b \in V'$, and $c, d, e, f \in V \setminus V'$, and $c\,\mathbf{EC}\,d$ and $e\,\mathbf{EC}\,f$. First assume that $a\,\mathbf{PO}\,b$, and that each of the relations between the regions a, b, c, d, e, f is **EC** or **PO** (i.e. **TPP** does not occur). Figure 1 illustrates how such networks can be weakly realized. Notice how the **EC** relations between c and d and between e and f are actually realized as **DC** relations. Furthermore, notice how the relations between (c, e), (c, f), (d, e), (d, f) can be independently chosen to be **DR** or **PO**. Since **TPP** relations can be realized as **TPP** or **NTPP**, scenarios where some relations are **TPP** can be weakly realized by straightforward variations of the situation in Figure 1 (left).

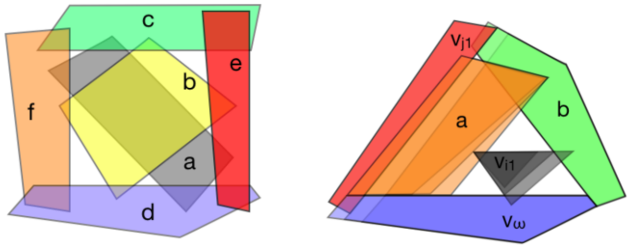


Figure 1. Weakly realizing a network where $a\,\mathbf{PO}\,b$, $c\,\mathbf{EC}\,d$, $e\,\mathbf{EC}\,f$ (left) and where $a\,\mathbf{PO}\,b$, $a\,\mathbf{EC}\,v_\omega$, $b\,\mathbf{EC}\,v_\omega$, and $v_{i_1}\,\mathbf{DC}\,v_{j_1}$ (right).

2.3 Arbitrary networks

Now we let Θ be an arbitrary, consistent basic network. However, without lack of generality, we can still assume that $R_{ij} \neq \mathbf{EQ}$ for $i \neq j$. Let $V' \cup \{v_{i_1}, ..., v_{i_k}\} \cup \{v_{j_1}, ..., v_{j_k}\}$ be a partition of V such that

1. $\Theta{\downarrow}V'$ is a network over $\{\mathbf{PO}, \mathbf{TPP}, \mathbf{TPP}^{-1}, \mathbf{EC}\}$;
2. $R_{i_l j_l} \in \{\mathbf{NTPP}, \mathbf{NTPP}^{-1}, \mathbf{DC}\}$ for every $l \in \{1, ..., k\}$.

From Section 2.2, we know that $\Theta{\downarrow}V'$ has an $(n-k)$-dimensional convex solution when $|V'| = 2n + 1 - 2k \geq 5$, i.e. $n - k \geq 2$. We will start from this solution, and, as in Section 2.2.1, repeatedly increment the number of regions that are considered, each time increasing the dimensionality of the solution. This will prove Theorem 1 for the case where $|V'| \geq 5$. The case where $|V'| \in \{1, 3\}$ is treated separately in Section 2.3.2.

2.3.1 Adding the regions $v_{i_{l+1}}$ and $v_{j_{l+1}}$

Assume that we have an $(n-k+l)$-dimensional convex solution $\mathcal{S}$ of $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_l}, v_{j_1}, ..., v_{j_l}\})$ with $0 \leq l \leq k-1$. Let us write $V^* = V' \cup \{v_{i_1}, ..., v_{i_l}, v_{j_1}, ..., v_{j_l}\}$. We now construct an $(n-k+l+1)$-dimensional convex solution $\mathcal{S}'$ of $\Theta{\downarrow}(V^* \cup \{v_{i_{l+1}}, v_{j_{l+1}}\})$. Let L and M be defined as in Section 2.2.1. We define the convex regions A and B in $\mathbb{R}^{n-k+l}$ as follows

$$A = L \cap \bigcap_{v_s \in V^*} \{er(\mathcal{S}(v_s)) \mid R_{i_{l+1}s} = \mathbf{NTPP}\}$$

$$B = M \cap \bigcap_{v_s \in V^*} \{er(\mathcal{S}(v_s)) \mid R_{j_{l+1}s} = \mathbf{NTPP}\}$$

where $er(X)$ represents an erosion of region X, defined as

$$er(X) = \{p \mid \forall q \,.\, (d(p, q) \leq \delta) \Rightarrow (q \in X)\}$$

with δ a sufficiently small constant and d the Euclidean distance.

We assume that $R_{i_{l+1}j_{l+1}} = \mathbf{DC}$; the cases where $R_{i_{l+1}j_{l+1}} \in \{\mathbf{NTPP}, \mathbf{NTPP}^{-1}\}$ are similar. Let $0 < a < e < f < g < b$ be fixed. We define

$$\mathcal{S}'(v_{i_{l+1}}) = \{(p_1, ..., p_{k'}, \lambda) \mid (p_1, ..., p_{k'}) \in A, \lambda \in [a, b]\}$$
$$\mathcal{S}'(v_{j_{l+1}}) = \{(p_1, ..., p_{k'}, \lambda) \mid (p_1, ..., p_{k'}) \in B, \lambda \in [-b, -a]\}$$

where we write k' for $n - k + l$. We say that a region v_m is at level r if there are regions $v_{m_1}, ..., v_{m_r}$ such that $R_{m_i m_{i+1}} = \mathbf{NTPP}$ for all i and $R_{m_r m} = \mathbf{NTPP}$, and there is no chain of $r + 1$ regions for which this is the case. We define for a region $v_s \in V^*$ at level r_s:

$$\mathcal{S}'(v_s) = \{(p_1, ..., p_{l+2}, \lambda) \mid (p_1, ..., p_{l+2}) \in \mathcal{S}(v_s), \lambda \in [v_s^-, v_s^+]\}$$

where

$$v_s^+ = \begin{cases} g & \text{If } R_{si_{l+1}} = \mathbf{TPP} \\ f + r_s \cdot \delta & \text{If } R_{si_{l+1}} = \mathbf{NTPP} \\ -a & \text{If } R_{sj_{l+1}} = \mathbf{TPP} \\ -e + r_s \cdot \delta & \text{If } R_{sj_{l+1}} = \mathbf{NTPP} \\ g + (r_s+1) \cdot \delta & \text{If } R_{si_{l+1}} = \mathbf{PO} \\ a & \text{If } R_{si_{l+1}} = \mathbf{EC} \\ (r_s+1) \cdot \delta & \text{If } R_{si_{l+1}} = \mathbf{DC} \\ & \quad \text{and } R_{sj_{l+1}} \notin \{\mathbf{TPP}, \mathbf{NTPP}\} \\ b & \text{If } R_{si_{l+1}} = \mathbf{TPP}^{-1} \\ b + (r_s+1) \cdot \delta & \text{If } R_{si_{l+1}} = \mathbf{NTPP}^{-1} \end{cases}$$

$$v_s^- = \begin{cases} a & \text{If } R_{si_{l+1}} = \mathbf{TPP} \\ e - r_s \cdot \delta & \text{If } R_{si_{l+1}} = \mathbf{NTPP} \\ -g & \text{If } R_{sj_{l+1}} = \mathbf{TPP} \\ -f - r_s \cdot \delta & \text{If } R_{sj_{l+1}} = \mathbf{NTPP} \\ -g - (r_s+1) \cdot \delta & \text{If } R_{sj_{l+1}} = \mathbf{PO} \\ -a & \text{If } R_{sj_{l+1}} = \mathbf{EC} \\ -(r_s+1) \cdot \delta & \text{If } R_{sj_{l+1}} = \mathbf{DC} \\ & \quad \text{and } R_{si_{l+1}} \notin \{\mathbf{TPP}, \mathbf{NTPP}\} \\ -b & \text{If } R_{sj_{l+1}} = \mathbf{TPP}^{-1} \\ -b - (r_s+1) \cdot \delta & \text{If } R_{sj_{l+1}} = \mathbf{NTPP}^{-1} \end{cases}$$

Lemma 5. *The mapping $\mathcal{S}'$ is an $(n-k+l+1)$-dimensional convex solution of $\Theta{\downarrow}(V' \cup \{v_{i_1}, ..., v_{i_{l+1}}, v_{j_1}, ..., v_{j_{l+1}}\})$.*

Proof. It is clear, by construction, that the relations $R_{i_{l+1}s}$ and $R_{j_{l+1}s}$ are satisfied by $\mathcal{S}'$ for all $v_s \in V^*$, and moreover that $R_{i_{l+1}j_{l+1}}$ is satisfied. We now show that also R_{st} is satisfied for all v_s and v_t in V^* (with $s \neq t$).

The relation $\mathbf{DC}$ is clearly preserved under the proposed "cylindrical" extension of $\mathcal{S}$ to $\mathcal{S}'$, even when $[v_s^-, v_s^+] \cap [v_t^-, v_t^+] \neq \emptyset$.

If $\mathcal{S}(v_s)\,\mathbf{EC}\,\mathcal{S}(v_t)$ holds, we will have $\mathcal{S}'(v_s)\,\mathbf{EC}\,\mathcal{S}'(v_t)$ as soon as $[v_s^-, v_s^+] \cap [v_t^-, v_t^+] \neq \emptyset$. If this were not the case, we would have that one of $R_{si_{l+1}}$, $R_{ti_{l+1}}$, $R_{sj_{l+1}}$ or $R_{tj_{l+1}}$ was either $\mathbf{TPP}$ or $\mathbf{NTPP}$. Assume for example $R_{si_{l+1}} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$ (the other cases are analogous). This implies that either $R_{ti_{l+1}} \notin \{\mathbf{DC}, \mathbf{EC}\}$ or both $R_{ti_{l+1}} = \mathbf{EC}$ and $R_{si_{l+1}} = \mathbf{TPP}$. In the former case, we have $\emptyset \neq [e, f] \subseteq [v_s^-, v_s^+] \cap [v_t^-, v_t^+]$, while in the second case we have $a \in [v_s^-, v_s^+] \cap [v_t^-, v_t^+]$.

If $\mathcal{S}(v_s)\,\mathbf{PO}\,\mathcal{S}(v_t)$ holds, we can show in an entirely analogous way that $\mathcal{S}'(v_s)\,\mathbf{PO}\,\mathcal{S}'(v_t)$.

If $\mathcal{S}(v_s)\,\mathbf{TPP}\,\mathcal{S}(v_t)$ holds, then we need to show that $v_s^- \geq v_t^-$ and $v_s^+ \leq v_t^+$. We show that $v_s^+ \leq v_t^+$; the proof for $v_s^- \geq v_t^-$ is entirely analogous. Note that $r_s \leq r_t$. If $R_{ti_{l+1}} = \mathbf{TPP}$, we have $v_t^+ = g$ and $R_{si_{l+1}} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$, hence either $v_s^+ = v_t^+ = g$ or $v_s^+ = f + r_s \cdot \delta < g = v_t^+$. If $R_{ti_{l+1}} = \mathbf{NTPP}$, we have $v_t^+ = f + r_t \cdot \delta$, $R_{si_{l_1}} = \mathbf{NTPP}$ and $v_s^+ = f + r_s \cdot \delta \leq f + r_t \cdot \delta = r_t^+$. If $R_{ti_{l+1}} = \mathbf{DC}$ and $R_{tj_{l+1}} \notin \{\mathbf{TPP}, \mathbf{NTPP}\}$, we have $R_{si_{l+1}} = \mathbf{DC}$ and $R_{sj_{l+1}} \notin \{\mathbf{TPP}, \mathbf{NTPP}\}$, and thus $v_s^+ = (r_s+1) \cdot \delta \leq (r_t+1) \cdot \delta = v_t^+$. If $R_{ti_{l+1}} = \mathbf{DC}$ and $R_{tj_{l+1}} = \mathbf{TPP}$, we have $v_t^+ = -a$, $R_{si_{l+1}} = \mathbf{DC}$, and $R_{sj_{l+1}} \in \{\mathbf{TPP}, \mathbf{NTPP}\}$, and thus $v_s^+ = -e + r_s \cdot \delta < -a = v_t^+$ or $v_s^+ = -a = v_t^+$. If $R_{ti_{l+1}} = \mathbf{DC}$ and $R_{tj_{l+1}} = \mathbf{NTPP}$, we have $v_t^+ = -e + r_t \cdot \delta$, $R_{si_{l+1}} = \mathbf{DC}$, and $R_{sj_{l+1}} = \mathbf{NTPP}$, and thus $v_s^+ = -e + r_s \cdot \delta \leq -e + r_t \cdot \delta = v_t^+$. If $R_{ti_{l+1}} = \mathbf{EC}$ we have $v_t^+ = a$, and $R_{si_{l+1}} \in \{\mathbf{DC}, \mathbf{EC}\}$, and thus $v_s^+ = (r_s+1) \cdot \delta < a = v_t^+$ or $v_s^+ = a = v_t^+$. If $R_{ti_{l+1}} = \mathbf{TPP}^{-1}$, we have $v_t^+ = b$.The only case where $v_s^+ > b$ could hold is when $R_{i_{l+1}s}$, which is impossible since $R_{st} = \mathbf{TPP}$. If $R_{ti_{l+1}} = \mathbf{NTPP}^{-1}$, we have $v_t^+ = b + r_t \cdot \delta$. The maximal value that v_s^+ could receive is $b + r_s \cdot \delta$ which cannot be greater than v_t^+ since $r_s \leq r_t$.

The case where $\mathcal{S}(v_s)\,\mathbf{NTPP}\,\mathcal{S}(v_t)$ holds is entirely analogous to the previous case. □

2.3.2 $|V'| \in \{1, 3\}$

Suppose $V' = \{v_\omega\}$. Take v_{i_1} and v_{j_1} from $V \setminus V'$. We observe that the network over $\{v_\omega, v_{i_1}, v_{j_1}\}$ has a convex solution in $\mathbb{R}$ as either $v_{i_1}\mathbf{NTPP}v_{j_1}$ or $v_{i_1}\mathbf{DC}v_{j_1}$.

Suppose $V' = \{a, b, v_\omega\}$. We observe that the network over $\{a, b, v_\omega\}$ has a convex solution in $\mathbb{R}$ except for the following cases:

- $a\,\mathbf{EC}\,b$, $b\,\mathbf{EC}\,v_\omega$, $a\,\mathbf{EC}\,v_\omega$.
- $a\,\mathbf{EC}\,b$, $a\,\mathbf{EC}\,v_\omega$, $b\,\mathbf{PO}\,v_\omega$.
- $a\,\mathbf{EC}\,b$, $a\,\mathbf{PO}\,v_\omega$, $b\,\mathbf{EC}\,v_\omega$.

In these cases, we need to show that a, b, v_ω can be realized in the plane together with two regions v_{i_1}, v_{j_1} from $V \setminus V'$, where $v_{i_1}\,\mathbf{NTPP}\,v_{j_1}$ or $v_{i_1}\,\mathbf{DC}\,v_{j_1}$. Assume for instance that $v_{i_1}\,\mathbf{DC}\,v_{j_1}$ and that no **PP** relations occur among $a, b, v_\omega, v_{i_1}, v_{j_1}$, then a scenario such as the one in Figure 1 (right) can be used, where some of the variations of the boundaries are shown, which could allow for variations of **DC**, **EC**, and **PO** relations. The remaining cases are similar.

3 Networks without convex solutions in n dimensions

Theorem 1 provides an upper bound on the number of dimensions for which we are guaranteed that networks with a given number of variables can be realized by convex regions. Although convex solutions may exist in lower-dimensional spaces for specific cases (e.g. networks of any size may be realizable by convex intervals), as we show in this section, for every number of dimensions n, there exist consistent networks which are not realizable using convex regions.

2 dimensions

Consider the RCC network Θ_{2D} defined by

$a\,\mathbf{EC}\,b$	$x\,\mathbf{TPP}\,a$	$y\,\mathbf{TPP}\,a$	$u\,\mathbf{TPP}\,b$
$v\,\mathbf{TPP}\,b$	$x\,\mathbf{DC}\,y$	$u\,\mathbf{DC}\,v$	$x\,\mathbf{EC}\,u$
$u\,\mathbf{EC}\,y$	$y\,\mathbf{EC}\,v$	$v\,\mathbf{EC}\,x$	

Clearly Θ_{2D} is consistent. However, it does not have any convex realizations in two dimensions. Indeed, if $\mathcal{S}$ were a two-dimensional convex solution, we clearly would have $\dim(\mathcal{S}(a) \cap \mathcal{S}(b)) \leq 1$. If $\dim(\mathcal{S}(a) \cap \mathcal{S}(b)) = 0$ then $\mathcal{S}(x)$, $\mathcal{S}(y)$, $\mathcal{S}(u)$ and $\mathcal{S}(v)$ could only meet in one point, which means that $x\,\mathbf{EC}\,u$,$u\,\mathbf{EC}\,y$ and $x\,\mathbf{DC}\,y$ could not be jointly satisfied. If $\dim(\mathcal{S}(a) \cap \mathcal{S}(b)) = 1$ then $\mathcal{S}(x) \cap \mathcal{S}(u)$, $\mathcal{S}(x) \cap \mathcal{S}(v)$, $\mathcal{S}(y) \cap \mathcal{S}(u)$ and $\mathcal{S}(y) \cap \mathcal{S}(v)$ are nonempty and pairwise disjoint. Take four points P_i $(i = 1, 2, 3, 4)$ from these sets and suppose $P_{j_1} < P_{j_2} < P_{j_3} < P_{j_4}$. We note that P_1 and P_2 are both in $\mathcal{S}(x)$, and P_3 and P_4 are both in $\mathcal{S}(y)$. Because x and y are disjoint, we know $\{P_1, P_2\} = \{P_{j_1}, P_{j_2}\}$ or $\{P_1, P_2\} = \{P_{j_3}, P_{j_4}\}$. Similarly, note that P_1 and P_3 are both in $\mathcal{S}(u)$, and P_2 and P_4 are both in $\mathcal{S}(v)$. Because u and v are disjoint, we know $\{P_1, P_3\} = \{P_{j_1}, P_{j_2}\}$ or $\{P_2, P_4\} = \{P_{j_3}, P_{j_4}\}$. This is a contradiction.

3 dimensions

Consider the network Θ_{3D} obtained by adding the following constraints to Θ_{2D}:

$$c\,\mathbf{EC}\,d \quad x\,\mathbf{TPP}\,c \quad y\,\mathbf{TPP}\,c \quad u\,\mathbf{TPP}\,d \quad v\,\mathbf{TPP}\,d$$
$$a\,\mathbf{PO}\,c \quad a\,\mathbf{PO}\,d \quad b\,\mathbf{PO}\,c \quad b\,\mathbf{PO}\,d$$

To see that Θ_{3D} is indeed not realizable in three dimensions, we show that $\dim(\mathcal{S}(a) \cap \mathcal{S}(b) \cap \mathcal{S}(c) \cap \mathcal{S}(d)) \leq 1$ for any three-dimensional convex solution $\mathcal{S}$, which leads to a contradiction as in the two-dimensional case.

Let H_1 be a hyperplane that separates $\mathcal{S}(a)$ and $\mathcal{S}(b)$, and let H_2 be a hyperplane that separates $\mathcal{S}(c)$ and $\mathcal{S}(d)$. This implies that $\mathcal{S}(a) \cap \mathcal{S}(b) \subseteq H_1$ and $\mathcal{S}(c) \cap \mathcal{S}(d) \subseteq H_2$. All we need to show is that $H_1 \neq H_2$. This is clear, however, because if $H_1 = H_2$ then H_1 would separate $\mathcal{S}(a)$ from $\mathcal{S}(c)$ or $\mathcal{S}(d)$, which means that $\mathcal{S}(a)$ could not partially overlap with both $\mathcal{S}(c)$ and $\mathcal{S}(d)$. Thus $\dim(H_1 \cap H_2) \leq 1$ which also means $\dim(\mathcal{S}(a) \cap \mathcal{S}(b) \cap \mathcal{S}(c) \cap \mathcal{S}(d)) \leq 1$.

n dimensions

For any $n \geq 4$ we consider the network Θ_{nD} obtained by adding the following constraints to Θ_{2D} for $i \in \{0, ..., n-3\}$

$$e_i\,\mathbf{TPP}\,a \quad e_i\,\mathbf{NTPP}\,f_i \quad a\,\mathbf{TPP}\,f_i \quad g_i\,\mathbf{EC}\,f_i$$
$$g_i\,\mathbf{EC}\,a \quad u\,\mathbf{TPP}\,g_i \quad v\,\mathbf{TPP}\,g_i \quad e_i\,\mathbf{EC}\,b$$
$$g_i\,\mathbf{TPP}\,b$$

and the following constraints for $i \in \{1, ..., n-3\}$

$$e_i\,\mathbf{EC}\,g_{i-1} \qquad g_i\,\mathbf{TPP}\,g_{i-1}$$

We show that $\dim(\mathcal{S}(a) \cap \mathcal{S}(g_{n-3})) \leq 1$ for any n-dimensional convex solution $\mathcal{S}$, which again leads to a contradiction, and thus that Θ_{nD} is not realizable by n-dimensional convex regions.

Let G_i be a hyperplane separating $\mathcal{S}(g_i)$ and $\mathcal{S}(f_i)$ for $i \in \{0, ..., n-3\}$, and let H_1 be a hyperplane separating $\mathcal{S}(a)$ and $\mathcal{S}(b)$ as before. We show by induction that $\dim(H_1 \cap G_0 \cap .... \cap G_k) \leq n-k-2$ for every $k \in \{0, ..., n-3\}$, from which the stated immediately follows.

First assume that $k = 0$. It suffices to show that $H_1 \neq G_0$ to show that $\dim(H_1 \cap G_0) \leq n-2$. If $H_1 = G_0$, we would have that $\mathcal{S}(a) \cap \mathcal{S}(b) \subseteq G_0$, and in particular that G_0 contains a boundary point of $\mathcal{S}(e_0)$; call this point P. However, since $\mathcal{S}(e_0)\,\mathbf{NTPP}\,\mathcal{S}(f_0)$, P would also belong to $\mathcal{S}(f_0)$, and since G_0 only contains boundary points of $\mathcal{S}(f_0)$, we would have that P is a boundary point of $\mathcal{S}(f_0)$ as well. This is a contradiction, since $e_0\,\mathbf{NTPP}\,f_0$ means that $\mathcal{S}(e_0)$ can only contain internal points of $\mathcal{S}(f_0)$.

For $k > 0$, we show that $H_1 \cap G_0 \cap ... \cap G_{k-1} \not\subseteq G_k$ in a similar way. Suppose $H_1 \cap G_0 \cap ... \cap G_{k-1} \subseteq G_k$ did hold. We have that $H_1, G_0, ..., G_{k-1}$ all separate $\mathcal{S}(a)$ from $\mathcal{S}(g_{k-1})$, hence $\mathcal{S}(a) \cap \mathcal{S}(g_{k-1}) \subseteq H_1 \cap G_0 \cap ... \cap G_{k-1} \subseteq G_k$. Since $\mathcal{S}(e_k) \subseteq \mathcal{S}(a)$ and $\mathcal{S}(e_k) \cap \mathcal{S}(g_{k-1}) \neq \emptyset$, there must exist a point P in $\mathcal{S}(e_k) \cap \mathcal{S}(g_{k-1})$ which is thus also in G_k. Clearly the point P is a boundary point of $\mathcal{S}(e_k)$, and since $e_k\,\mathbf{NTPP}\,f_k$, we have that P is an internal point of $\mathcal{S}(f_k)$. This is a contraction, since G_k was assumed to be a hyperplane separating $\mathcal{S}(f_k)$ from $\mathcal{S}(g_k)$.

For any number of dimensions n we can thus find an RCC network which is consistent but cannot be realized by convex n-dimensional regions. The counterexamples we have provided for two and three dimensions are optimal, in the sense that they involve $2n+2$ regions, i.e. any convex network with fewer regions would necessarily be realizable by convex regions in n dimensions. The counterexample for $n \geq 4$ dimensions, on the other hand, uses $3n$ regions, and the question whether counterexamples with fewer regions exist currently remains open.

4 Conclusions

Although restricting to convex solutions to RCC8 networks is known to be a demanding requirement in low-dimensional Euclidean spaces, we have shown that it does not affect the notion of consistency in Euclidean spaces of a sufficiently high dimension. Specifically, we have shown that consistent RCC8 networks of up to $2n+1$ variables can always be realized by using convex regions in $\mathbb{R}^n$. We have furthermore shown that this bound is optimal for 2 and 3 dimensions, and that for $n \geq 4$ there exist consistent RCC networks of $3n$ variables which are not realizable in n dimensions using convex regions. This not only furthers our understanding of the notion of consistency in RCC8, but also enables the use of existing RCC8 decision procedures for reasoning about qualitative descriptions of conceptual spaces.

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