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Hard and Easy k-Typed Compact Coalitional Games: The Knowledge of Player Types Marks the Boundary

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Abstract. Coalitional games model scenarios where rational agents can form coalitions so as to obtain higher worths than by acting in isolation. Once a coalition forms and obtains its worth, the problem of how this worth can be fairly distributed has to be faced. Desirable worth distributions are usually referred to as solution concepts. Recent research pointed out that, while reasoning problems involving such solution concepts are hard in general for games specified in compact form (e.g., graph games), some of them, in particular the core, become tractable when agents come partitioned into a fixed number k of *types*, i.e., of classes of strategically equivalent players. The paper continues along this line of research, by firstly showing that two other relevant solution concepts, the *kernel* and the *nucleolus*, are tractable in this setting and independently of the specific game encoding, provided worth functions are given as a polynomial-time computable oracles. Then, it analyzes a different setting where games are still k-typed but the actual player partitioning is not a-priori known. Within this latter setting, the paper addresses the question about how efficiently strategic equivalence between pairs of players can be recognized, and reconsiders the computational complexity of the core, the kernel, and the nucleolus. All such problems and notions emerged to be intractable, thereby evidencing that the knowledge of player types marks the boundary of tractability for reasoning about k-typed coalitional games.

1 Introduction

Coalitional games have been adopted by the AI community as useful formal tools to analyze cooperative behavior. Once a coalition forms and obtains its worth, one has to face the problem of how this worth can be fairly distributed. Several solution concepts, such as the *core*, the *kernel*, and the *nucleolus* (see, e.g., [12]), have been introduced and thoroughly studied through the years with the aim of characterizing fair worth distributions.

Looking at players' decision processes about worth distributions, it is sensible to assume players' reasoning resources not to come unbounded and to use the tools of computational complexity as a viable mean to model and reason about this bounded rationality principle. In particular, it is easily noted that computational questions are of interest whenever the function specifying the worth associated with each possible coalition is encoded in some succinct way, e.g., when it is given in terms of polynomially computable functions over some combinatorial structure. Indeed, all problems trivialize if we explicitly represent the entire extent of a worth function, which requires exponential space in the number of involved players. Coalitional games whose worth functions are encoded by means of some succinct representation mechanism will be hereinafter called *compact games*.

Problem	$\mathcal{C}$(FP)	$\mathcal{C}_k$(FP)$\wedge$TBF	$\mathcal{C}_k$(FP)*
IN-CORE	co-NP-c [8]	in P [15, 1]	co-NP-c
CORE-NONEMPT.	co-NP-c [8]	in P [15, 1]	co-NP-c
IN-KERNEL	Δ_2^{P}-c [8]	in P	co-NP-h
IN-NUCLEOLUS	Δ_2^{P}-c [7]	in P	co-NP-h
NUCLEOLUS-COMP.	FΔ_2^{P}-c [7]	in FP	NP-h

Figure 1. Summary of results. $\mathcal{C}$(FP): games having polynomial-time worth functions; $\mathcal{C}_k$(FP): games in $\mathcal{C}$(FP) with k player-types at most; TBF: type-based form, where types are given. *Using randomized reductions.

Unfortunately, large part of the complexity analysis carried out on compact coalitional games undebatably demonstrated that computing with most of the aforementioned solution concepts is intractable in general. This emerges from the first column of the table reported in Figure 1, where IN-X denotes the problem of deciding membership in the solution concept X, CORE-NONEMPTINESS is the problem of deciding the non-emptiness of the core, and NUCLEOLUS-COMPUTATION is the problem of computing the nucleolus. There, note that hardness results have been shown for specific compact game settings (in particular, *graph games* [5] and *marginal contribution nets* [9]), while membership results hold over the whole class $\mathcal{C}$(FP) of all those games whose worth functions are computable via polynomial-time FP oracles (see [8, 7]). As a matter of fact, however, all these results deal with settings where each player in the game may have a distinctive behavior.

On the contrary, it is everyday life experience that people (and agents!), in reasoning within a specific decision context, behave according to some (sometimes, few) behavioral schemas, which are often known in advance to the scenario analyst. For instance, in many applications agents are naturally clustered according to technological features (e.g., they model mobile phones sharing data in a wireless network, and are classified according to bandwidth and energetic features). Therefore, it is often the case that we have a large number of agents, but in fact they belong to a limited number of categories, usually called *types*, that determine their behavior in the game at hands. Being this setting natural to many practical contexts and useful, it is sensible to ask whether, or to which extent, the complexity of reasoning with solution concepts for a given class of coalitional games is influenced by knowing that the number of players' types is small, formally, is bounded by some fixed constant.

This is precisely the perspective introduced by Shrot et al. [14], who defined the setting and mainly focused on graph games and games with *synergies among coalitions* [4], and then put forward by Ueda et al. [15] and by Aadithya at al. [1], who extended the

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analysis to arbitrary classes of games with FP worth functions: Let $\mathcal{C}_k(\text{FP}) \subset \mathcal{C}(\text{FP})$ be the subclass of $\mathcal{C}(\text{FP})$ of all those games whose players can be partitioned into at most k types, with k being a fixed natural number, and let us say that a game in $\mathcal{C}_k(\text{FP})$ is in *type-based* form if the type of each player is known a-priori. Then, our current knowledge is that IN-CORE and CORE-NONEMPTINESS are feasible in polynomial time over games in $\mathcal{C}_k(\text{FP})$ that are moreover given in type-based form [15, 1]. In fact, extending the analysis to further solution concepts has been left as an open research issue [15].

In this paper, we start by addressing the above research issue, and our first contribution is to completely characterize the complexity of the kernel and the nucleolus. Indeed,

- ▷ We show that IN-KERNEL, IN-NUCLEOLUS, and NUCLEOLUS-COMPUTATION are all feasible in polynomial time over games in $\mathcal{C}_k(\text{FP})$ that are given in type-based form (see the second column in Figure 1). Note that the nucleolus is always guaranteed to be non-empty (whenever some imputation exists) and to be a single point contained in the kernel [12]. Thus, our results immediately entail that, in the given setting, a point in the kernel can also be computed efficiently.

Note that the above tractability results assume that player types are known a-priori. While this is certainly the case in many practical scenarios, one might naturally wonder whether tractability results still hold if we know that players have a limited number of types, but we do not know how they are actually partitioned, i.e., we do not know the type of each player. We address these questions too:

- ▷ First, we focus on the basic problem of deciding whether two players have the same type, and we show that it is intractable, formally co-NP-complete over games in $\mathcal{C}(\text{FP})$. Note that we already know from the literature [14] that the problem is intractable over games with synergies among coalitions. However, the result is hardly surprising given that such games are unlikely in $\mathcal{C}(\text{FP})$, as the associated worth function is already NP-hard to compute [4].
- ▷ Then, we consider the problem of recognizing whether a game in $\mathcal{C}(\text{FP})$ is actually in $\mathcal{C}_k(\text{FP})$, and we show that this is intractable as well (co-NP-complete).

Motivated by the above bad news, we eventually consider a kind of "mixed" setting, where games actually belong to $\mathcal{C}_k(\text{FP})$, but they are not given in type-based form. That is, we know the maximum number k of distinct types in any game of the class, but we do not know the type of each player. Even under the given promise, intractability results still emerge, thereby evidencing that the knowledge of player types marks the boundary of tractability for reasoning about k-typed coalitional games. In particular:

- ▷ We show that deciding whether two players have the same type is co-NP-complete over games in $\mathcal{C}_k(\text{FP})$ (not in type-based form), where hardness holds under *randomized reductions* (see [16]). On this class and under the same complexity model, computing the number of distinct player types is shown to be intractable too.
- ▷ We reconsider all computation problems related to the core, the kernel, and the nucleolus, and we show that they are intractable (under randomized reductions) on the class $\mathcal{C}_k(\text{FP})$ for games that are not given in type-based form (see the third column in Figure 1).

Organization. Section 2 introduces the setting and the framework of k-typed coalitional games. The analysis of games given in type-based form is reported in Section 3. Computational issues related to the problem of finding the actual partitioning of the players are discussed in Section 4, while the setting where games in $\mathcal{C}_k(\text{FP})$ are not given in type-based form is studied in Section 5.

2 Formal Framework

In this section we recall some basic notions about game theory and introduce the classes of games considered in the following.

2.1 Coalitional Games

A coalitional game $\mathcal{G}$ is a pair $\langle N, v\rangle$, where N is the set of all the players and $v : 2^N \mapsto \mathbb{R}$ is the worth function. A vector $(x_i)_{i\in N}$ (with $x_i \in \mathbb{R}$) is an *imputation* of $\mathcal{G}$ if $\sum_{i\in N} x_i = v(N)$ and $x_i \geq v(\{i\})$, for each $i \in N$. In the following, for an imputation x and a coalition $S \subseteq N$, we denote by $x(S)$ the value $\sum_{i\in S} x_i$. The set of all the imputations of $\mathcal{G}$ is denoted by $X(\mathcal{G})$. Several *solution concepts* have been proposed to characterize the most desirable imputations of coalitional games. Below, we recall the notions of *core*, *kernel*, and *nucleolus* (see, e.g., [12]).

Core. The *core* $\mathscr{C}(\mathcal{G})$ of a coalitional game $\mathcal{G} = \langle N, v\rangle$ is the set of all imputations x that are "stable", for there is no coalition whose members may receive a higher payoff than in x by leaving the grand-coalition: $\mathscr{C}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists S \subseteq N$ and $(y_i)_{i\in S}$ such that $y(S) = v(S)$ and $y_k > x_k, \forall k \in S\}$.

Kernel. For any pair of players i and j of $\mathcal{G}$, let $\mathcal{I}_{i,j}$ be the set of all coalitions containing player i but not player j. The *excess* of a coalition S at $x \in X(\mathcal{G})$, denoted by $e(S,x)$, is defined as $v(S) - x(S)$. The *surplus* $s_{i,j}(x)$ of i against j at x is $s_{i,j}(x) = \max_{S\in\mathcal{I}_{i,j}} e(S,x)$. Then, the *kernel* $\mathscr{K}(\mathcal{G})$ of a game $\mathcal{G} = \langle N, v\rangle$ is the set: $\mathscr{K}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\}), \forall i,j \in N, i \neq j\}$.

Nucleolus. For any imputation x of $\mathcal{G}$, we define the vector: $\theta(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^n-1},x))$, where the various excesses of all coalitions (but the empty one) are arranged in non-increasing order. Let $\theta(x)[i]$ denote the i-th element of $\theta(x)$. For a pair of imputations x and y, we say that $\theta(x)$ is *lexicographically smaller* than $\theta(y)$, denoted by $\theta(x) \prec \theta(y)$, if there exists a positive integer q such that $\theta(x)[i] = \theta(y)[i]$ for all $i < q$ and $\theta(x)[q] < \theta(y)[q]$. Then, the *nucleolus* $\mathscr{N}(\mathcal{G})$ of a game $\mathcal{G}$ is the set $\mathscr{N}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid \nexists y \in X(\mathcal{G})$ s.t. $\theta(y) \prec \theta(x)\}$.

Relationships. It is well-known that, for any coalitional game $\mathcal{G}$ with $X(\mathcal{G}) \neq \emptyset$, $|\mathscr{N}(\mathcal{G})| = 1$; $\mathscr{N}(\mathcal{G}) \subseteq \mathscr{K}(\mathcal{G})$ (hence, $\mathscr{K}(\mathcal{G}) \neq \emptyset$); and if $\mathscr{C}(\mathcal{G}) \neq \emptyset$, then $\mathscr{N}(\mathcal{G}) \subseteq \mathscr{C}(\mathcal{G})$ (see, e.g., [12]).

2.2 k-Typed Games

Guided by the observation that obstructions to tractability of coalitional games often emerge in scenarios where most players are "different", Shrot et al. [14] recently re-considered several problems for coalitional games, by studying their computational complexity by taking the number of distinct player *types* as a parameter.

Formally, given a coalitional game $\mathcal{G} = \langle N, v\rangle$, Shrot et al. [14] define two players $i, j \in N$ as *strategically equivalent* in $\mathcal{G}$ (or, simply, having the same *type*) if $v(S\cup\{i\}) = v(S\cup\{j\})$ holds, for each coalition $S \subseteq N$ such that $S \cap \{i,j\} = \emptyset$. Then, a coalitional game is said to be *k-typed* if its players can be partitioned into k classes of pairwise strategically equivalent players. The intuition is that a number of intractable problems related to solution concepts of compact coalitional games might be efficiently solved on classes of k-typed games, whenever k is some fixed natural number.

2.3 Computational Setting and Representations

Complexity Classes. The class P (resp., NP) is the set of *decision* problems solvable by a deterministic (resp., non-deterministic) Turing machine in polynomial time, that is, in time $||x||^{O(1)}$, where $||x||$

denotes the size of the input x. The class of problems whose complementary problems are in NP is co-NP. Moreover, Δ_2^{P} is the class of problems solvable in polynomial time by a deterministic machine using an NP Turing machine as an oracle. To capture the complexity of *computation* problems, we consider instead *deterministic transducers*, i.e., deterministic Turing machines T equipped with a write-only output tape. Then, denote by FP (resp., $\mathrm{F}\Delta_2^{\mathrm{P}}$) the class of all functions that can be computed by a deterministic transducer in polynomial time (resp., and by using an NP Turing machine as an oracle).

Game Representation. We assume that the input for any decision problem consists of a game $\mathcal{G} = \langle N, v\rangle$, and that the game representation includes the list of players, so that, for every coalition $S \subseteq N$, $||S|| \leq ||\mathcal{G}||$ holds. We say that $\mathcal{G}$ is an FP-game if the worth function belongs to FP. The class of all FP-games is denoted by $\mathcal{C}(\mathrm{FP})$.

Well-known classes of FP-games are *graph* and *hypergraph games* [5], *marginal contribution nets* [9], *games in multi-issue domains* [3], and *weighted voting games* [2]. For further compact representations schemes for coalitional games, we refer the interested reader to the classification described in [8].

3 Complexity Analysis of k-Typed Games

Recall that on arbitrary FP-games, the core, the kernel, and the nucleolus are intractable solution concepts as evidenced in Figure 1. Our interest here is to re-consider these concepts over FP-games, where the number of distinct types is bounded by some fixed natural number k. Formally, for any fixed natural number k, let $\mathcal{C}_k(\mathrm{FP})$ be the class of all FP-games that are furthermore k-typed.

In particular, as commonly done in the literature, a coalitional k-typed game $\mathcal{G}$ is viewed in this section as a tuple $\langle (N_1, \ldots, N_k), v\rangle$, where $N_1, \ldots, N_k$ are disjoint sets of players, with all players in N_i having the same type. In this case, we say that $\mathcal{G}$ is given in *type-based* form. In fact, note that in this setting, one may always assume that the worth function is given in the form $v_t : \{1, \ldots, |N_1|\} \times \cdots \times \{1, \ldots, |N_k|\} \mapsto \mathbb{R}$, which is the kind of worth functions studied in [15, 1]. Indeed, this trivially follows by the result below.

Proposition 3.1 ([14]). *Let $\langle (N_1, \ldots, N_k), v\rangle$ be a k-typed game. Given any two coalitions $S, T \subseteq N_1 \cup \cdots \cup N_k$, if $|S \cap N_i| = |T \cap N_i|$, for each $i \in \{1, \ldots, k\}$, then $v(S) = v(T)$.*

In this paper, for notational uniformity, we prefer to use "standard" worth functions, and to exploit instead a subset of all possible coalitions spanning v: Assume that an arbitrary ordering of players in N is fixed, and define the *characteristic-coalitions set* $\mathcal{D}_{\mathcal{G}} \subseteq 2^N$ as the set of coalitions $\{(P_1 \cup P_2 \cup \cdots \cup P_k) \subseteq N \mid S \subseteq N$, and P_i contains the first $|S \cap N_i|$ players from set N_i, $1 \leq i \leq k\}$. Note that the size of $\mathcal{D}_{\mathcal{G}}$ is polynomial w.r.t. the size of $\mathcal{G}$, as it contains at most $|N_1| \times |N_2| \times \cdots \times |N_k|$ coalitions.

On the class $\mathcal{C}_k(\mathrm{FP})$, if games are given in type-based form, IN-CORE and CORE-NONEMPTINESS are in P [15, 1]. In the rest of the section, we extend the analysis to other relevant solution concepts.

3.1 Nucleolus

We start the analysis with the nucleolus. In this case, it is relevant to characterize the "structure" of this solution concept over k-typed coalitional games. The following result shows that the nucleolus is in fact "symmetric" w.r.t. player types.

Theorem 3.2. *Let $\mathcal{G} = \langle N, v\rangle$ be coalitional game, and let x be the unique imputation in $\mathscr{N}(\mathcal{G})$. Then, $x_i = x_j$ holds, for each pair of players i and j in N having the same type.*

Proof. Assume by contradiction that there are two players i and j in N having the same type and such that $x_i \neq x_j$ (in particular, w.l.o.g., such that $x_i > x_j$). We claim that $\{x\} \neq \mathscr{N}(\mathcal{G})$.

Let x' be the worth assignment where the values assigned to i and j are swapped, that is, such that $x'_i = x_j$, $x'_j = x_i$, and $x'_p = x_p$, for each $p \in N \setminus \{i, j\}$. Note that, for any coalition S such that $S \cap \{i, j\} = \emptyset$ or $\{i, j\} \subseteq S$, the total worth does not change, and hence $e(S, x) = s(S, x')$. It remains to consider all pairs of symmetric coalitions T, T' such that $i \in T$ and $j \notin T$, $i \notin T'$ and $j \in T'$, and with all other elements being the same, i.e., $T \setminus \{i, j\} = T' \setminus \{i, j\}$. Note that for each $p \in T \cap T'$, $x_p = x'_p$, and that $v(T) = v(T')$ as i and j have the same type. It follows that, for every such pair of coalitions, $e(T', x') = e(T, x)$ and $e(T, x') = e(T', x)$; that is, their excesses are just swapped. Therefore, the vector of excesses does not change when considering x' in place of x, and we get $\theta(x) = \theta(x')$, which is impossible because $|\mathscr{N}(\mathcal{G})| = 1$. □

For the sake of completeness, note that the converse of Theorem 3.2 does not hold. For instance, on the game $\mathcal{G}_0 = \langle \{a, b, c\}, v_0\rangle$ such that $v_0(\{a\}) = v_0(\{b\}) = v_0(\{c\}) = 1$, $v_0(\{a, b, c\}) = 3$, $v_0(\{a, b\}) = 1$, $v_0(\{a, c\}) = 2$, and $v_0(\{b, c\}) = 3$, the vector x with $x_a = x_b = x_c = 1$ is the only imputation and hence belongs to $\mathscr{N}(\mathcal{G}_0)$, but the three players have different types.

Computation. With the above result in place, let us focus on the problem of computing the nucleolus. Let $\mathcal{G} = \langle N, v\rangle$ be a game, and consider the following linear programming problem $\mathtt{LP}_t$, for $t > 0$:

$$
\begin{aligned}
\mathtt{LP}_t = \{\min \varepsilon \mid\ & x(S) = v(S) - \varepsilon_r, \quad \forall S \in \Lambda_r, \forall 1 \leq r \leq t-1 \\
& x(S) \geq v(S) - \varepsilon, \quad \forall S \subseteq N \\
& x \in \Omega\},
\end{aligned}
$$

where Ω is a convex subset of $\mathbb{R}^N$; ε_r is the optimum value of the program $\mathtt{LP}_r$ evaluated at the r-th step; and $\Lambda_r = \{S \subseteq N \mid x(S) = v(S) - \varepsilon_r, \forall x \in V_r\}$ with $V_r = \{x \mid (x, \varepsilon_r)$ is an optimal solution to $\mathtt{LP}_r\}$ is the set of all coalitions having exactly excess ε_r on all the optimal solutions of the program $\mathtt{LP}_r$.

By [11] (see also [6]), it is known that there is an index t_* such that $\mathtt{LP}_{t_*}$ has exactly one optimal solution (x_*, ϵ_{t_*}), and $\theta(x_*) \prec \theta(x)$ holds, for any $x \in \Omega$. In particular, $\{x_*\} = \mathscr{N}(\mathcal{G})$, whenever Ω is the set $X(\mathcal{G})$ of all imputations for $\mathcal{G}$. Moreover, it is known that the approach, with an adjustment discussed in [7], provides a $\mathrm{F}\Delta_2^{\mathrm{P}}$ membership result for computing the nucleolus on games in $\mathcal{C}(\mathrm{FP})$. A corresponding Δ_2^{P}-hardness result is obtained even for the underlying decision problem IN-NUCLEOLUS on graphical games [7]. Below, we show that the problem is no longer intractable on the class $\mathcal{C}_k(\mathrm{FP})$, if player types are known.

Theorem 3.3. *On the class $\mathcal{C}_k(\mathrm{FP})$, if games are given in type-based form, then* NUCLEOLUS-COMPUTATION *is in* FP.

Proof Sketch. Let $\mathcal{G} = \langle (N_1, \ldots, N_k), v\rangle$ be a k-typed coalitional game, and consider the convex set $\widehat{X}(\mathcal{G}) = \{x \in X(\mathcal{G}) \mid x_i = x_j$, for each pair i,j of players having the same type$\}$. By Theorem 3.2, $\mathscr{N}(\mathcal{G}) \subseteq \widehat{X}(\mathcal{G})$, and thus $\mathscr{N}(\mathcal{G})$ can be computed by the above sequence of linear programs by setting $\Omega = \widehat{X}(\mathcal{G}) \subseteq X(\mathcal{G})$ (see Lemma 6.5 in [11]). In fact, having restricted the feasible regions of these programs to $\widehat{X}(\mathcal{G})$, it follows that every inequality associated with some coalition S entails every other inequality obtained by replacing any variable x_i (associated with a player) of a certain type by any other variable x_j (associated with a player) of the same type. As a consequence, it is sufficient to consider only inequalities associated with the coalitions in the characteristic set $\mathcal{D}_{\mathcal{G}}$, in place of all subsets of N.

Thus, in order to compute the nucleolus of $\mathcal{G}$, instead of using $\mathtt{LP}_t$, we build the following sequence of linear programming problems:

$$\widehat{\mathtt{LP}}_t = \{\min \varepsilon \mid x(S) = v(S) - \varepsilon_r, \quad \forall S \in \Lambda_r, \forall 1 \leq r \leq t-1 \\ x(S) \geq v(S) - \varepsilon, \quad \forall S \in \mathcal{D}_{\mathcal{G}} \\ x \in \widehat{X}(\mathcal{G})\},$$

where ε_r is the optimum value of the program $\widehat{\mathtt{LP}}_r$ evaluated at the r-th step, and $\Lambda_r = \{S \in \mathcal{D}_{\mathcal{G}} \mid x(S) = v(S) - \varepsilon_r, \forall x \in V_r\}$ with $V_r = \{x \mid (x, \varepsilon_r)$ is an optimal solution to $\widehat{\mathtt{LP}}_r\}$.

Note that any linear program in the above sequence contains just polynomially many distinct inequalities. We next show that such programs can be also computed and solved in polynomial time.

Let us start with the first program $\widehat{\mathtt{LP}}_1$, which consists only of inequalities associated with coalitions in $\mathcal{D}_{\mathcal{G}}$ (there are no equalities). Because $\mathcal{G}$ is in type-based form, all these inequalities may be computed in polynomial time by iterating over all possible combinations of numbers of players per type. Thus, by standard results in mathematical programming [13], the optimum value ε_1 of $\widehat{\mathtt{LP}}_1$ can be computed in polynomial time.

Then, in order to build $\widehat{\mathtt{LP}}_2$, we have to build the set Λ_1 (the set of all coalitions from $\mathcal{D}_{\mathcal{G}}$ having exactly excess ε_1 on the optimal solutions of $\mathtt{LP}_1$). Note that a coalition $\bar{S}$ belongs to Λ_1 if and only if the set $\{x \in \widehat{X}(\mathcal{G}) \mid x(S) \geq v(S) - \varepsilon_1, \forall S \in \mathcal{D}_{\mathcal{G}}$, and $x(\bar{S}) > v(\bar{S}) - \varepsilon_1\}$ is empty, and this condition can be checked in polynomial time. Thus, $\widehat{\mathtt{LP}}_2$ can be built in polynomial time.

Eventually, we can inductively apply the method above to construct $\widehat{\mathtt{LP}}_t$, for each $t > 0$. Concerning the number of iterations, note that, at each step t, at least one coalition from $\mathcal{D}_{\mathcal{G}}$ enters in Λ_t. Thus, after at most polynomially many steps the process converges to the nucleolus, as the size of $\mathcal{D}_{\mathcal{G}}$ is polynomial w.r.t. the size of $\mathcal{G}$. □

As the nucleolus is a singleton set, we immediately obtain the following corollary.

Corollary 3.4. *On the class* $\mathcal{C}_k$(FP)*, if games are given in type-based form, then* IN-NUCLEOLUS *is in* P.

3.2 Kernel

Theorem 3.5. *On the class* $\mathcal{C}_k$(FP)*, if games are given in type-based form, then* IN-KERNEL *is in* P.

Proof Sketch. Recall the definition of the kernel. Notice that we have to verify the condition $s_{i,j}(x) > s_{j,i}(x) \Rightarrow x_j = v(\{j\})$, for all distinct players i and j of N. Thus, if computing the surplus $s_{i,j}(x)$ is feasible in polynomial time, then the whole procedure can be carried out in polynomial time. We claim that, in fact, this is the case.

Let $\mathcal{G} = \langle (N_1, \ldots, N_k), v \rangle$ be a k-typed game, and recall that $s_{i,j}(x) = \max_{S \in \mathcal{I}_{i,j}} e(S, x)$, where $e(S, x) = v(S) - x(S)$. Let $n_1, \ldots, n_k$ be the number of players in $N_1, \ldots, N_k$, respectively. So, we can rewrite the surplus as follows:

$$s_{i,j}(x) = \max_{\substack{0 \leq t_p \leq n_p,\ \forall p: 1 \leq p \leq k \\ \text{s.t. } t_1 + \cdots + t_k \geq 1}} \quad \max_{\substack{S \in \mathcal{I}_{i,j} \\ |S \cap N_q| = t_q,\ \forall q: 1 \leq q \leq k}} \quad v(S) - x(S).$$

Because of Proposition 3.1, note that $v(S) = v(T)$ holds, for each pair of coalitions S and T such that $|S \cap N_i| = |T \cap N_i| = t_i$, for each $i \in \{1, \ldots, k\}$. However, the imputation x might be such that $\{(v(S) - x(S)) \mid S \subseteq N_1 \cup \cdots \cup N_k\}$ contains exponentially many distinct values, as x is not necessarily a symmetric one. This problem can be circumvented by exploiting the clustering of the players in their types. Indeed, for each cluster N_i, we sort its players based on the ascending values of the worth they receive in x. Hence, we can compute the term

$$\max_{\substack{S \in \mathcal{I}_{i,j} \\ |S \cap N_q| = t_q,\ \forall q: 1 \leq q \leq k}} \quad v(S) - x(S)$$

by simply evaluating $v(S) - x(S)$ on the specific coalition S containing, for each cluster N_i, the first t_i players w.r.t. such order, by always including i and excluding j. By this, the first maximization requires iterating over polynomially many elements, and for each of them the above polynomial-time method can be exploited to compute the value of the subsequent term. Thus, the whole procedure can be carried out in polynomial time in the number of players. □

3.3 Specific Classes of Compact Games

We conclude the section by noticing that, as a corollary of the above general results, we can get the tractability of well-known classes of games whose worth functions are computable in FP, and for which determining player types is feasible in polynomial time. Recall that, for any fixed k, a k-typed graph game or game with synergies among coalitions can be represented in type-based form (i.e., the clustering of its players can be found) in polynomial time [14]. In fact, given the type-based form for such kinds of games, IN-CORE and CORE-NONEMPTINESS can be solved in polynomial time, too [15, 1]. Below, we complete the picture with the other solution concepts.

Corollary 3.6. *For any fixed k, on k-typed games given as graph games or games with synergies among coalitions,* IN-KERNEL*,* IN-NUCLEOLUS*, and* NUCLEOLUS-COMPUTATION *are in* P.

4 On The Hardness of Finding Player Types

In [14], it has been observed that deciding whether two players have the same type in *games with synergies among coalitions* [4] is an NP-hard problem—as discussed above, the problem is instead tractable if the number of agent types is fixed by a constant k. In fact, this NP-hardness result is hardly surprising given that such games are unlikely FP-games, as the associated worth function is NP-hard to compute [4]. Hence, the intrinsic difficulty of the worth function actually obscures the complexity of the problem defined on top of it. Our first result is to strengthen this analysis, by showing that the problem remains intractable even on FP-games. In particular, we shall show that the problem is complete for the class co-NP.

Before stating the result, we fix some definitions that will be used in the following. For any Boolean formula ϕ over a set X of variables, we define the FP-game $\mathcal{G}_\phi = \langle X, v_\phi \rangle$, whose players coincide with the variables in ϕ, and where, for each coalition $S \subseteq X$,

$$v_\phi(S) = \begin{cases} 1, \text{ if } \sigma(S) \models \phi, \text{ i.e., } \sigma(S) \text{ is a satisfying assignment} \\ 0, \text{ otherwise,} \end{cases}$$

with $\sigma(S)$ denoting the truth assignment where a variable x_i evaluates *true* if and only if the corresponding player x_i belongs to S.

Moreover, consider the following problem *Critical Swap* (CS): Given a tuple $\langle \phi, x_i, x_j \rangle$, where ϕ is a Boolean formula over a set X of variables and $\{x_i, x_j\} \subseteq X$, decide whether $\{x_i, x_j\}$ is a *critical pair* (w.r.t. ϕ), i.e., decide whether there is a satisfying truth assignment $\bar{\sigma}$ such that: (1) $\bar{\sigma}[x_i] \neq \bar{\sigma}[x_j]$ and (2) the assignment σ', where $\sigma'[x_k] = \bar{\sigma}[x_k]$, for each $x_k \in X \setminus \{x_i, x_j\}$, $\sigma'[x_i] = \bar{\sigma}[x_j]$, and $\sigma'[x_j] = \bar{\sigma}[x_i]$, is not satisfying. It is easy to see that CS *is* NP-*hard*, by a reduction from SAT: For any Boolean formula γ, let $\phi = \gamma \wedge x_a \wedge \neg x_b$ be a new Boolean formula where x_a and x_b are fresh variables (i.e., not in γ). It is immediate to check that γ is satisfiable if and only if $\langle \phi, x_a, x_b \rangle$ is a "yes" instance of CS.

Theorem 4.1. *On the class $\mathcal{C}$(FP), deciding whether two players have the same type is co-*NP*-complete.*

Proof Sketch. Consider the complementary problem of deciding whether two players p and q do not have the same type. We show that the problem is NP-complete. Membership in NP is easily seen, as we can guess a coalition S with $S \cap \{p, q\} = \emptyset$, and then check in polynomial time that $v(S \cup \{p\}) \neq v(S \cup \{q\})$.

Hardness is next proven via a reduction from problem CS. Let ϕ be a Boolean formula over a set X of variables with $\{x_i, x_j\} \subseteq X$, and let us build in polynomial time the game $\mathcal{G}_\phi = \langle X, v_\phi \rangle$.

We show that $\langle \phi, x_i, x_j \rangle$ *is a "yes" instance of* CS $\Leftrightarrow$ x_i *and* x_j *do not have the same type in* $\mathcal{G}_\phi$.

($\Rightarrow$) Let $\bar{\sigma}$ be an assignment witnessing that $\langle \phi, x_i, x_j \rangle$ is a "yes" instance. Assume, w.l.o.g., that $\bar{\sigma}[x_i] = \mathit{true}$ and $\bar{\sigma}[x_j] = \mathit{false}$. Let $\overline{S} \subseteq X$ be the coalition such that $\sigma(\overline{S}) = \bar{\sigma}$, and note that $x_i \in \overline{S}$ and $x_j \notin \overline{S}$. Consider the coalition $T = \overline{S} \setminus \{x_i\}$, hence such that $\sigma(T \cup \{x_i\}) \models \phi$. By definition of a solution to CS, $\sigma(T \cup \{x_j\}) \not\models \phi$. It follows that $v_\phi(T \cup \{x_i\}) = 1$ while $v_\phi(T \cup \{x_j\}) = 0$. Thus, x_i and x_j do not have the same type.

($\Leftarrow$) Assume that $\langle \phi, x_i, x_j \rangle$ is a "no" instance. We consider two cases. (1) ϕ *is unsatisfiable*. In this case, $v_\phi(S) = 0$ holds, for each coalition $S \subseteq X$, and x_i and x_j have trivially the same type. (2) ϕ *is satisfiable*. In this case, for each set $T \subseteq X \setminus \{x_i, x_j\}$, we have that either $\sigma(T \cup \{x_i\}) \not\models \phi$ and $\sigma(T \cup \{x_j\}) \not\models \phi$, or $\sigma(T \cup \{x_i\}) \models \phi$ and $\sigma(T \cup \{x_j\}) \models \phi$. Hence, $v_\phi(T \cup \{x_i\}) = v_\phi(T \cup \{x_j\})$ holds, and x_i and x_j have the same type. □

The above is very bad news, but it does not immediately imply that determining whether the number of player types is bounded by some given constant is an intractable problem. Our second result is to characterize the complexity of this problem.

Theorem 4.2. *On the class $\mathcal{C}$(FP), deciding whether a game is k-typed is a co-*NP*-complete problem. Hardness holds even for $k = 1$.*

Proof Sketch. We show that deciding whether there are at least $k' = k + 1$ player types is NP-complete. For the membership, it suffices to guess a set P of k' players together with $k'(k' - 1)/2$ coalitions, and then check in polynomial time that such coalitions witness that players in P are pairwise not strategically equivalent.

For the hardness part, consider the problem *Exists Critical Swap (*ECS*)*, in which given a Boolean formula ϕ over a set X of variables, we have to decide whether there exists a critical pair $\{x_i, x_j\}$ w.r.t. ϕ. It is easily seen that ECS is NP-hard. Indeed, for any Boolean formula γ, let $\phi = \gamma \wedge x_a \wedge \neg x_b$ be a new Boolean formula where x_a and x_b are fresh variables. Then, γ is satisfiable if and only if $\langle \phi \rangle$ is a "yes" instance of ECS.

Our result then follows by showing that: ϕ *is a "yes" instance of* ECS $\Leftrightarrow$ $\mathcal{G}_\phi$ *has at least two players with different type (hence $k > 1$).*

($\Rightarrow$) Assume that x_i and x_j are two variables in X such that $\langle \phi, x_i, x_j \rangle$ is a "yes" instance of CS. By the same line of reasoning as in the proof of Theorem 4.1, we have that x_i and x_j are not strategically equivalent, and hence in $\mathcal{G}_\phi$ there are at least 2 different types of players.

($\Leftarrow$) Assume now that, for each pair of variables x_i and x_j of ϕ, the tuple $\langle \phi, x_i, x_j \rangle$ is a "no" instance of CS. In the case where ϕ is unsatisfiable, $v_\phi(S) = 0$ holds, for each coalitions S. Hence, every player in $\mathcal{G}_\phi$ have the same type. Consider then the case where ϕ is satisfiable, but there is no critical pair $\{x_i, x_j\}$ w.r.t. ϕ. In this latter case, for any chosen pair x_i and x_j, we can apply the same line of reasoning as in the proof of Theorem 4.1 (case (2) of the ($\Leftarrow$)-part), and conclude that x_i and x_j are strategically equivalent. As this holds for each pair of players, we have that all players have the same type. □

5 Shedding Lights on The Grey Area

So far, we have shown tractability results for the class $\mathcal{C}_k$(FP) where games are given in type-based form, and we have pointed out that deciding whether a game is actually in $\mathcal{C}_k$(FP) is an intractable problem. Our analysis has thus still a missing piece: What happens if a game is known to belong to $\mathcal{C}_k$(FP), but it is not given in type-based form (i.e., with player types being actually unknown)? In this section, the question will be addressed.

5.1 On the Hardness of Bounded-Types Games

Our first result is to show that identifying player types is likely intractable even on the class $\mathcal{C}_k$(FP) of games that actually have such a bounded number of types. The proofs of intractability results are based here on a complexity-theory setting developed to study problems that are believed to be difficult but could not be classified using the most common reductions (i.e., Karp or Turing reductions).

Consider the problem SAT_1, where we have to decide the satisfiability of a Boolean formula ϕ, under the promise that ϕ admits *at most one* satisfying assignment. This is the prototypical NP-hard problem *under randomized reductions* [16]. It is widely believed that such problems are not feasible in polynomial time. For our aims here, it is not necessary to expand on the concept of randomized reductions, and we refer the interested reader, for instance, to [10]. Indeed, the promise of dealing with a fixed number of player types is next related to SAT_1 via "standard" reductions from this problem, in order to prove the analogue of Theorem 4.1 and Theorem 4.2 for classes of games having bounded types.

Theorem 5.1. *On the class $\mathcal{C}_k$(FP), deciding whether a game is k'-typed, for any constant k' with $k' < k$, and whether two players have the same type are co-*NP*-complete under randomized reductions. Hardness holds even for $k = 2$.*

Proof Sketch. Membership results in co-NP follows by Theorem 4.1 and Theorem 4.2. Concerning the hardness part, we exhibit a polynomial-time reduction from SAT_1. Let ϕ' be a Boolean formula over the set X' of variables having one satisfying assignment at most, and define $\phi = \phi' \wedge x_\alpha \wedge \neg x_\beta$ as a Boolean formula over the set $X = X' \cup \{x_\alpha, x_\beta\}$. Note that ϕ has one satisfying assignment at most, where in particular x_α (resp., x_β) evaluates to true (resp., false). Consider the associated game $\mathcal{G}_\phi$, and observe that if ϕ is unsatisfiable, then $v_\phi(S) = 0$ holds, for each coalition $S \subseteq X$. Thus, in this case, there is only one type of players, and $\mathcal{G}_\phi$ is 1-typed.

Assume now that $\widetilde{\sigma}$ is the satisfying truth assignment for ϕ. Let $\widetilde{S}$ be the coalition such that $\sigma(\widetilde{S}) = \widetilde{\sigma}$, and let x_i and x_j be two arbitrary players. Then, two cases have to be considered:

(1) Assume that x_i and x_j are two players such that $x_i \in \widetilde{S}$ and $x_j \notin \widetilde{S}$. Consider the coalition $\widetilde{T} = \widetilde{S} \setminus \{x_i\}$, and note that $v_\phi(\widetilde{T} \cup \{x_i\}) = 1$ and $v_\phi(\widetilde{T} \cup \{x_j\}) = 0$. Hence, x_i and x_j have two different types.

(2) Assume that either $\{x_i, x_j\} \subseteq \widetilde{S}$ or $\{x_i, x_j\} \cap \widetilde{S} = \emptyset$. Let T be any coalition such that $\{x_i, x_j\} \cap T = \emptyset$. We claim that $v_\phi(T \cup \{x_i\}) = 0$ and $v_\phi(T \cup \{x_j\}) = 0$ hold. Indeed, first observe that $\widetilde{T} \cup \{x_i\} \neq \widetilde{S}$ and $\widetilde{T} \cup \{x_j\} \neq \widetilde{S}$. Then, the claim follows by just noticing that $\widetilde{S}$ is the one coalition for which $v_\phi(\widetilde{S}) = 1$. Hence, in this case, x_i and x_j have the same type.

By combining the above two cases, we have that players of $\mathcal{G}_\phi$ can be partitioned into exactly two different strategic types: Players in $\widetilde{S}$, and players outside $\widetilde{S}$. Therefore, $\mathcal{G}_\phi$ is 2-typed, but it is not 1-typed. It follows that $\mathcal{G}_\phi$ is 1-typed if and only if ϕ (and, hence, the original formula ϕ') is unsatisfiable. This shows that deciding whether a game is 1-typed is co-NP-complete under randomized reductions.

Finally, in order to show that deciding whether two players have the same type is co-NP-complete under randomized reductions, it suffices to observe that x_α and x_β have the same type if and only if ϕ (and, hence, ϕ') is unsatisfiable. □

5.2 Complexity of Solution Concepts

Now, we turn to the analysis of the complexity of solution concepts. Figure 1 reports the intrinsic difficulty of various reasoning problems involving the core, the kernel, and the nucleolus on the class $\mathcal{C}(\text{FP})$. All results are intractability ones. Here, we complete the picture, by showing that focusing on the class $\mathcal{C}_k(\text{FP})$ does not guarantee their tractability. We start with problems related to the core.

Theorem 5.2. *On the class $\mathcal{C}_k(\text{FP})$, the problems* IN-CORE *and* CORE-NONEMPTINESS *are co-*NP*-complete under randomized reductions. Hardness holds even for $k = 2$.*

Proof Sketch. For both problems membership in co-NP follows by the results for the larger class $\mathcal{C}(\text{FP})$ (see Figure 1). For the hardness of IN-CORE, consider the reduction in the proof of Theorem 5.1 based on the Boolean formulae ϕ' over variables in X', and ϕ over $X = X' \cup \{x_\alpha, x_\beta\}$. Let z be a vector mapping each player to 0, and note that $z(X) = v_\phi(X) = 0$ (recall here that in order to have $v_\phi(S) > 0$, it is required that $x_\beta \notin S$). Then, $z \in \mathscr{C}(\mathcal{G}_\phi)$ if and only if for each $S \subseteq X$, $v_\phi(S) = 0$. By definition of the worth function, this latter holds if and only if ϕ (hence ϕ') is not satisfiable. From this observation, we easily get the result for CORE-NONEMPTINESS, too. Indeed, just recall that $v_\phi(X) = 0$ holds for the grand-coalition X, and hence the above vector z is the only one that might in principle belong to $X(\mathcal{G}_\phi)$ (as all worth values are non-negative). Thus, $\mathscr{C}(\mathcal{G}_\phi) \neq \emptyset$ if and only if $z \in \mathscr{C}(\mathcal{G}_\phi)$, which completes the proof.

Note that, in the proof, we can even assume w.l.o.g. that ϕ' is such that $v_\phi(S) = 0$, for each S with $|S| = 1$, thereby showing that hardness holds even if z is guaranteed to be an imputation. □

We now continue with the decision problems related to the nucleolus and the kernel. Note that in the results below, the corresponding membership results are missing.

Theorem 5.3. *On the class $\mathcal{C}_k(\text{FP})$,* IN-KERNEL *and* IN-NUCLEOLUS *are co-*NP*-hard under randomized reductions. Hardness holds even for $k = 2$.*

Proof Sketch. Consider again the reduction in the proof of Theorem 5.1 based on the Boolean formulae ϕ' over variables in X' and ϕ over $X = X' \cup \{x_\alpha, x_\beta\}$. Define a new game $\bar{\mathcal{G}}_\phi = \langle X, \bar{v}_\phi \rangle$ where $\bar{v}_\phi(X) = 1$ and $\bar{v}_\phi(S) = v_\phi(S)$, for each $S \subset X$. Let z be the imputation assigning the worth $1/|X|$ to each player.

First, we claim that $z \in \mathscr{K}(\bar{\mathcal{G}}_\phi)$ holds if and only if ϕ is not satisfiable. Indeed, if ϕ is not satisfiable, then $\bar{v}_\phi(S) = 0$, for each $S \subset X$. Hence, for each pair of players x_p and x_q, $s_{x_p,x_q}(z) = s_{x_q,x_p}(z) = -1/|X|$, and hence z is in $\mathscr{K}(\bar{\mathcal{G}}_\phi)$. On the other hand, if ϕ is satisfiable, then there is a coalition S (with $x_\alpha \in S$ and $x_\beta \notin S$) such that $\bar{v}_\phi(S) = 1$. Thus, we have that $s_{x_\alpha,x_\beta}(z) = 1 - 1/|X| > s_{x_\beta,x_\alpha}(z) = -1/|X|$. However, $\bar{v}_\phi(\{x_\beta\}) = 0 \neq 1/|X|$, and hence $z \notin \mathscr{K}(\bar{\mathcal{G}}_\phi)$.

We complete the picture by claiming that $z \in \mathscr{N}(\bar{\mathcal{G}}_\phi)$ holds if and only if ϕ is not satisfiable. Indeed, if ϕ is not satisfiable, then $\bar{v}_\phi(S) = 0$, for each $S \subset X$, and it can be easily checked that symmetrically distributing the worth of $\bar{v}_\phi(X)$ over all players leads to the nucleolus. Instead, if ϕ is satisfiable, then there is a coalition S (with $x_\alpha \in S$ and $x_\beta \notin S$) such that $\bar{v}_\phi(S) = 1$. Consider the imputation z' where each player in S (resp., outside S) gets worth $1/|S|$ (resp., 0). Then, $\theta(z') \prec \theta(z)$, and hence $z \notin \mathscr{N}(\bar{\mathcal{G}}_\phi)$. □

A simple corollary of the complexity of IN-NUCLEOLUS is the following characterization for the computation problem.

Corollary 5.4. *On the class $\mathcal{C}_k(\text{FP})$,* NUCLEOLUS-COMPUTATION *is* NP*-hard under randomized reductions, even for $k = 2$.*

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