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# An Axiom System for a Spatial Logic with Convexity

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**Abstract.** This paper presents a part of work in progress on axiomatizing a spatial logic with convexity and inclusion predicates (hereinafter called convexity logic), with some intended interpretation over the real plane. More formally, let $\mathbf{L}_{conv,\leq}$ be a language of first order logic and two non-logical primitives: *conv* (interpreted as a property of a set of being convex) and $\leq$ (interpreted as the set inclusion relation). We let variables range over regular open rational polygons in the real plane (denoted $ROQ(\mathbf{R}^2)$). We call the tuple $\mathbf{M} = \langle ROQ, conv, \leq \rangle$ — where primitives are defined as indicated above — a standard model. We propose an axiomatization of the theory of $\mathbf{M}$ and prove soundness and completeness for this axiomatization.

## 1 Introduction

In recent years there has been a considerable amount of attention devoted to the theories with geometrical interpretation, called as *spatial logics* (e.g. [6]). Recently, there has been a considerable amount of attention devoted to *topological* spatial logics. This research stems from the practical motivations found in the AI community, where the idea of *qualitative* (non-numerical) spatial reasoning is being developed ([7]). This research program also embraces the *point-free* approach to spatial logics, as it greatly reduces the computational costs. The region connection calculus (RCC-8) is a prominent example (see [8]). RCC-8 features 8 basic *topological* relations between regions in a topological space. For many practical applications however, purely topological logics are too weak to provide an adequate description of relevant spatial information ([1], [4]). In this paper we present a part of work in progress on axiomatizing an *affine* spatial logic with convexity and inclusion predicates (hereinafter called *convexity logic*, see below for details, [5]). In 1959 Alfred Tarski published a paper entitled *What is Elementary Geometry?* ([10]). He described there a first-order theory with a *geometrical* interpretation. Tarski's investigations began a new chapter in the quest to formalise geometry, initiated in Euclid's *The Elements*. The novelty of Tarski's approach lies in changing the focus from geometry itself to the *language* that describes it, which allowed him to use the full apparatus of formal logic and model theory. We follow this approach here. It is important to realise that even slight alterations in considered interpretations can affect the computational and model theoretical properties of considered logics. It is standard to assume that regions are either regular open or closed subsets of some topological space. The choice between open and closed regular sets does not in most cases have much of an effect - the resulting spatial logics would be equivalent (e.g. Duntsch in the case of RCC-8). Considering, however — as it is done in the literature, cf. [7] — one of the possible refinements of the domain can affect the resulting logic ([9]). In our case setting the variables to range over regular open polygons instead of regular open rational polygons does affect the resulting logic. The convexity logics of regular open polygons and regular open rational polygons are different ([5]). Here we investigate the convexity logic with variables ranging over regular open rational polygons of the real plane. This logic turns out surprisingly expressive, to the point where we can (locally) reestablish the coordinate system. It is known that this logic is undecidable ([5]) — however it might still be worthwhile to axiomatise it, thus making it fit for possible use in KR&R environment. Note that its propositional fragment is decidable but as computationaly hard as the satisfiability problem for the theory of reals ([4]). Throrough this paper we assume familiarity with basic model theoretic notions. When in doubt, the reader is invited to consult [3]. In the sequel, knowledge of basic topology and very basic affine geometry is assumed. Consult [2] for any affine geometry notion not explained here. Finally, thourough this paper $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$ denote the sets of natural, integer and real numbers, respectively. $\mathbf{R}^2$ denotes the real plane.

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## 2 Formalisation

Let $\mathbf{L}_{conv,\leq}$ be the first-order language with two primitive predicates: binary $\leq$ and unary *conv*; and two constant symbols: 0 and 1. We work with a first order theory over $\mathbf{L}_{conv,\leq}$, where $\leq$ is interpreted as the inclusion relation and *conv* as a property of a set of being convex. As usual in region-based spatial logics, variables are set to range over certain subsets of $\mathbf{R}^2$. We start with defining a notion of a *regular open* set.

**Definition 1.** *Let $S$ be a subset of some topological space. We denote the* interior *of $S$ by $S^0$ and the* closure *of $S$ by $S^-$. $S$ is called* regular open *if $S = (S)^{-0}$.*

The following result is standard.

**Theorem 1.** *The set of regular open sets in $X$ forms a Boolean algebra $RO(X)$ with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $x \cdot y = x \cap y$, $x + y = (x \cup y)^{-0}$ and $-x = (X/x)^0$.*

We follow the usual practice and restrict our attention to certain *well behaved* regular open sets ([7], p. 14-21). Note that every line in $\mathbf{R}^2$ divides $\mathbf{R}^2$ into two residual domains, called *half-planes*. Half-planes are regular open, hence we can speak about the sums, products and complements of half-planes in $RO(\mathbf{R}^2)$. By a *regular open rational polygon* we mean a Boolean combination in $RO(\mathbf{R}^2)$ of finitely many half-planes bounded by lines with rational coefficients in $\mathbf{R}^2$. We denote the set of all regular open rational polygons in

$\mathbf{R}^2$ by $ROQ(\mathbf{R}^2)$. Note that $ROQ(\mathbf{R}^2)$ is a Boolean subalgebra of $RO(\mathbf{R}^2)$.

**Definition 2.** *A non-empty set $S \in \mathbf{R}^2$ is called* convex *if, for all $(\zeta_1, \zeta_2), (\zeta_1', \zeta_2') \in S$ and for all $\alpha \in [0,1]$ we have:*

$$(\alpha \cdot \zeta_1 + (1-\alpha)\cdot \zeta_1', \alpha \cdot \zeta_2 + (1-\alpha)\cdot \zeta_2') \in S.$$

*The empty set is taken to be non-convex.*

Having defined the domain of discourse, let us set up the $\mathbf{L}_{conv,\leq}$-structure it gives rise to.

**Definition 3.** *We define the model $\mathbf{M}$ to have the domain $ROQ(\mathbf{R}^2)$ and the following interpretations of the predicates in $\mathbf{L}_{conv,\leq}$.*

$\leq^{\mathbf{M}} = \{\langle a, b\rangle \in ROQ(\mathbf{R}^2) : a \subseteq b\}$;

$conv^{\mathbf{M}} = \{a \in ROQ(\mathbf{R}^2) : a \text{ is convex}\}$;

$0^{\mathbf{M}} = \emptyset$;

$1^{\mathbf{M}} = \mathbf{R}^2$.

*We sometimes refer to $\mathbf{M}$ as the* rational model.

If $\phi$ is a formula, $\phi(x_1, \ldots, x_n)$ means that $\phi$ has at most $n$ variables: $x_1, \ldots, x_n$. If an $n$-tuple of regions $a_1, \ldots, a_n$ satisfy $\phi$ in $\mathbf{M}$, we write $\phi[a_1, \ldots, a_n]$.

## 2.1 Basic results

In what follows we introduce some more notational conventions as well as provide a number of basic results. Most of these are self-explanatory and are therefore left without proofs. This section for the most part contains results from [5].

**Theorem 2.** *Let $l \in ROQ(\mathbf{R}^2)$. Then $\mathbf{M} \models hp[l]$ if and only if $l$ is a half-plane, where $hp(x)$ is the formula $conv(x) \wedge conv(-x)$.*

*Proof.* It is enough to observe that for any convex region $l$ its complement $-l$ is also convex if and only if $l$ is a half-plane. □

**Theorem 3.** *Let $l_1, l_2 \in ROQ(\mathbf{R}^2)$. Then $l_1$ and $l_2$ are half-planes with lines bounding them being coincident if and only if $\mathbf{M} \models \alpha[l_1, l_2]$, where $\alpha(x_1, x_2)$ is the formula $\bigwedge_{1\leq i\leq 2} hp(x_i) \wedge (x_1 = x_2 \vee x_1 = -x_2)$.*

*Proof.* Clearly, two lines are coincident just in case they bound the same half-planes. □

Observe that for any two half-planes $l_1, l_2$ the bounding lines of $l_1$ and $l_2$ are parallel if and only if at least one of $l_1 \cdot l_2 = \emptyset$, $l_1 \cdot -l_2 = \emptyset$, $-l_1 \cdot l_2 = \emptyset$, $-l_1 \cdot -l_2 = \emptyset$ holds.

**Theorem 4.** *Let $l_1, l_2 \in ROQ(\mathbf{R}^2)$. Then there exists a formula $par(x_1, x_2)$ such that $\mathbf{M} \models par[l_1, l_2]$ if and only if $l_1$ and $l_2$ are half-planes and lines bounding them are parallel.*

**Definition 4.** *Let $l_1, l_2, l_3$ be any non-parallel, non-coincident lines with $l_1 \cap l_2 = \mathbf{O}$, $l_1 \cap l_3 = \mathbf{I}$ and $l_2 \cap l_3 = \mathbf{J}$. We say that $l_1, l_2, l_3$ form a* coordinate system *and call $l_1$ the* abscissa, *$l_2$ the* ordinata *and refer to point $\mathbf{O}$ as the origin and to segments $\overline{\mathbf{OI}}$ and $\overline{\mathbf{OJ}}$ as the units of measurement on the lines they belong to.*

For the sake of simplicity, we sometimes identify a given half-plane with the line bounding it. To add clarity, we introduce the following convention: let $l$ and $-l$ be two half-planes bounded by the same line. We always take $-l$ to be on the lefthandside of this line. (Technically this is done by considering which half-plane contains $\mathbf{O}$.)

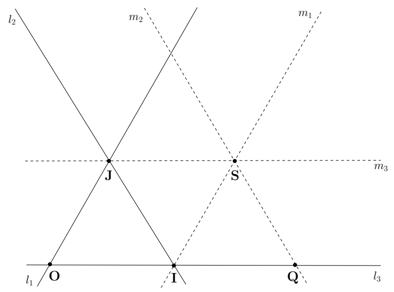


**Figure 1.** Lines $l_1$,$l_2$,$l_3$ form a coordinate system. By the construction described in theorem 9: $\overline{\mathbf{OI}} = \overline{\mathbf{OQ}}$.

For an example coordinate system see figure 1. The above definition allows us to fix our notation and terminology for the sake of clarity. It should be obvious however that the enumeration of lines does not really matter. All the results to follow apply if, for example, we chose to think of $l_2$ or $l_1$ as abscissa. This is used, in fact, in the proofs of some of the theorems presented here.

**Definition 5.** *A* general line pair *(GLP) is a pair $[l, m]$ such that $l, m \in ROQ(\mathbf{R}^2)$ are half-planes whose bounding lines intersect at a unique point **P**. We call **P** the point* determined *by $[l, m]$.*

**Theorem 5.** *Let $l_1, l_2 \in ROQ(\mathbf{R}^2)$. There exists a formula $[x_1, x_2]$ such that $\mathbf{M} \models [l_1, l_2]$ if and only if $l_1, l_2$ are half-planes and lines bounding them form a general line pair.*

**Theorem 6.** *Let $l_1, l_2, l_3, l_4 \in ROQ(\mathbf{R}^2)$. There exists a formula $[x_1, x_2] \doteq [x_3, x_4]$ such that $\mathbf{M} \models [l_1, l_2] \doteq [l_3, l_4]$ if and only if $l_1, l_2$ and $l_3, l_4$ are half planes such that lines bounding them form general line pairs which determine the same point.*

**Theorem 7.** *There exists a formula $coor(x_1, x_2, x_3)$ such that $\mathbf{M} \models coor(l_1, l_2, l_3)$ if and only if $l_1, l_2, l_3$ are half-planes such that lines bounding them form a coordinate system.*

**Theorem 8.** *Let $l_1, l_2, l_3$ be rational lines forming a coordinate system with points $\mathbf{O}, \mathbf{I}, \mathbf{J}$ as defined above. Let $m_1, m_2, m_3$ be rational lines such that the following conditions all hold: (i) for each $l_i$ and $m_i$: $l_i \parallel m_i$, (ii) $m_1 \cap m_2 \cap m_3 = \mathbf{S}$, (iii) $l_1 \cap l_2 \cap m_3 = \mathbf{J}$, (iv) $l_2 \cap l_3 \cap m_1 = \mathbf{I}$, (v) $l_3 \cap m_2 = \mathbf{Q}$; then $\overline{\mathbf{OI}} = \overline{\mathbf{OQ}}$ (as shown in figure 1).*

**Theorem 9.** *Let $l_1, l_2, l_3, m \in ROQ(\mathbf{R}^2)$ be half-planes and let $l_1, l_2, l_3$ form a coordinate frame. There exists a formula $\phi_n(x_1, x_2, x_3, y)$ such that for any rational line $m$ intersecting the line bounding $l_3$ at a point $\mathbf{K}$, $\mathbf{M} \models \phi_n[l_1, l_2, l_3, m]$ if and only if $\overline{\mathbf{OK}} = n\overline{\mathbf{OI}}$, where $n$ is a natural number.*

*Proof.* Clearly, construction from Theorem 9 is expressible in $\mathbf{L}_{\leq,conv}$. We obtain the desired result by repeating this construction several times. □

**Theorem 10.** *Theorem 10 holds when $n$ is replaced by a rational number $q = \frac{n}{m}$.*

**Theorem 11.** *Let $l_1, l_2, l_3, m$ be half-planes in $ROQ(\mathbf{R}^2)$, such that $\mathbf{M} \models coord[l_1, l_2, l_3]$. There exists a formula $\tau(x_1, x_2, x_3)$ such that if $\mathbf{M} \models \tau[l_1, l_2, l_3, m]$ then, for any half-plane $m'$ $\mathbf{M} \models \tau[l_1, l_2, l_3, m']$ if and only if $m = m'$.*

To see how the proof proceeds, note that given a rational line $l$ there are eight possible (exclusive) arrangements of $l$ in reference to a given coordinate frame formed by $l_1, l_2, l_3$. Our task is to formalise each of these in our language. Let $m, n, p, q \in \mathbf{Z}, q, n \neq 0$.

1. $l$ is coincident with $l_1$ - formalised $\mathbf{M} \models \tau_{<1,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<1,m,n,p,q>}(x_1, x_2, x_3, x) := \alpha(x, x_1)$;
2. $l$ is coincident with $l_2$ - formalised $\mathbf{M} \models \tau_{<2,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<2,m,n,p,q>}(x_1, x_2, x_3, x) := \alpha(x, x_2)$;
3. $l$ is coincident with $l_3$ - formalised $\mathbf{M} \models \tau_{<3,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<3,m,n,p,q>}(x_1, x_2, x_3, x) := \alpha(x, x_3)$;
4. $l$ is parallel to $l_3$ and intersects $l_1$ at some point $\mathbf{P}$ - formalised $\mathbf{M} \models \tau_{<4,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<4,m,n,p,q>}(x_1, x_2, x_3, x) := \exists z(\phi_m(x_3, z, x_1, x_2) \wedge \phi_n(x_3, z, x_1, x)) \wedge par(x, x_3)$;
5. $l$ is parallel to $l_1$ and intersects $l_3$ at some point $\mathbf{P}$ - formalised $\mathbf{M} \models \tau_{<5,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<5,m,n,p,q>}(x_1, x_2, x_3, x) := \exists z(\phi_m(x_1, z, x_3, x_2) \wedge \phi_n(x_1, z, x_3, x)) \wedge par(x, x_1)$;
6. $l$ intersects $l_1$ and $l_3$ at the same point ($\mathbf{O}$) and instersects $l_2$ at some point $\mathbf{P}$ - formalised $\mathbf{M} \models \tau_{<6,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<6,m,n,p,q>}(x_1, x_2, x_3, x) := \exists z(\phi_m(x_3, z, x_2, x_1] \wedge \phi_n[x_3, z, x_2, x]) \wedge (x, x_1) \doteq (x, x_3)$;
7. $l$ intersects $l_1$ and $l_3$ at the same point ($\mathbf{O}$) and is parallel to $l_2$ - formalised $\mathbf{M} \models \tau_{<7,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<7,m,n,p,q>}(x_1, x_2, x_3, x) := \phi_0(x_1, x_2, x_3, x) \wedge par(x, x_2)$;
8. $l$ intersects $l_1$ and $l_3$ at some points $\mathbf{P}$ and $\mathbf{Q}$ respectively - formalised $\mathbf{M} \models \tau_{<8,m,n,p,q>}[l_1, l_2, l_3, l]$ where $\tau_{<8,m,n,p,q>}(x_1, x_2, x_3, x) := \exists z(\phi_m(x_1, z, x_3, x_2) \wedge \phi_n(x_1, z, x_3, x)) \wedge \exists y(\phi_q(x_3, y, x_1, x_2) \wedge \phi_p(x_3, y, x_1, x))$.

We refer to any formula $\tau$ used above as a *fixing formula*. We avoid superscripts where they are inessential.The notion of a fixing formula is crucial here as it allows us to prove the completeness theorem for our axiomatisation (see below).

We finish this section with a result justifying the name *affine* logic that in the introduction was associated with convexity logic.

**Definition 6.** *A mapping $\tau : \mathbf{R}^2 \to \mathbf{R}^2$ is an* affine transformation *if it is of the form $\tau((\zeta_1, \zeta_2)) = (\zeta_1, \zeta_2)M + (\alpha, \beta)$, where $M$ is a nonsingular matrix and $(\alpha, \beta) \in \mathbf{R}^2$. If, in addition, $\alpha$, $\beta$ and the elements of $M$ are all rational, we say that $\tau$ is rational affine.*

**Definition 7.** *Two $n$-tuples of regions $\bar{a}$ and $\bar{b}$ in $ROP(\mathbf{R}^2)$ are said to be* affine equivalent*, written $\bar{a} \sim \bar{b}$, if there is an affine transformation $\tau$ taking $\bar{a}$ to $\bar{b}$.*

**Definition 8.** *A formula $\phi(\bar{x})$ is said to be* affine complete *in $\mathbf{M}$ if any two $n$-tuples in $ROP(\mathbf{R}^2)$ satisfying $\phi$ in $\mathbf{M}$ are affine-equivalent.*

One of the most important results in [5] is the following. For any two $n$-tuples of regions in $\mathbf{M}$, if these are affine equivalent then they satisfy the same formulas in $\mathbf{M}$. Conversely, every $n$-tuple of regions satisfies some affine-complete formula.

Note that this result relies heavily on the notion of fixing formula.

## 3 Axioms

Our aim is to axiomatise $\mathbf{L}_{conv,\leq}$-theory of $\mathbf{M}$, denoted $Th(\mathbf{M})$. Let $y = bc(x_1, \ldots, x_n)$ be any formula of the form $y = \sum_{I \in P} \prod_{i \in I} x_i$, where $S = \{1, \ldots, n\}$ and $P \subseteq 2^S$ such that for every $i \in S$ there exists $I \in P$ such that $i \in I$. We call $y = bc(x_1, \ldots, x_n)$ a *Boolean combination* formula.

Let $K \neq \emptyset$ and let $m, m', n, n', p, p', q, q' \in \mathbf{Z}$ with $n, n', q, q' \neq 0$. We propose the following axiomatisation of $Th(\mathbf{M})$:

1. axioms of non-trivial Boolean Algebra;
2. $\exists x_1 \exists x_2 \exists x_3 coord(x_1, x_2, x_3)$;
3. $\neg conv(0)$;
4. $\forall x_1 \ldots \forall x_n \forall y((\bigwedge_{i \in S} hp(x_i) \wedge y = bc(x_1, \ldots, x_n)) \to (conv(y) \leftrightarrow \bigvee_{K \subseteq S} \prod_{k \in K} x_k = y))$;
   where *bc* is a Boolean combination formula.
5. $\forall x_1 \forall x_2 \forall x_3 \forall y(\bigwedge_{1 \leq i \leq m} \tau_i(x_1, x_2, x_3, y_i) \to \prod_{1 \leq i \leq m} y_i = 0)$, where any element of $ROQ(\mathbf{R}^2)$ bounded by the half-planes fixed by $\tau_i$ in reference to any coordinate system is empty;
6. $\forall y_1 \ldots \forall y_m \forall x_1 \forall x_2 \forall x_3 (\tau_{<k,m,n,p,q>}(x_1, x_2, x_3, y) \to \neg \tau_{<k',m',n',p',q'>}(x_1, x_2, x_3, y))$,
   where $k \neq k'$;
7. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau_{<k,m,n,p,q>}(x_1, x_2, x_3, y) \to \neg \tau_{<k,m',n',p',q'>}(x_1, x_2, x_3, y))$,
   where $k \in \{5, 6, 7\}$ and $\frac{m}{n} \neq \frac{m'}{n'}$;
8. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau_{<8,m,n,p,q>}(x_1, x_2, x_3, y) \to \neg \tau_{<8,m',n',p',q'>}(x_1, x_2, x_3, y))$,
   where $\frac{m}{n} \neq \frac{m'}{n'}$ or $\frac{p}{q} \neq \frac{p'}{q'}$;
9. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau_{<k,m,n,p,q>}(x_1, x_2, x_3, y) \wedge \tau_{<k,m',n',p',q'>}(x_1, x_2, x_3, y')) \to y = y'$,
   where $k \in \{1, 2, 3, 4\}$;
10. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau_{<k,m,n,p,q>}(x_1, x_2, x_3, y) \wedge \tau_{<k,m',n',p',q'>}(x_1, x_2, x_3, y')) \to y = y'$,
    where $k \in \{5, 6, 7\}$ and $\frac{m}{n} = \frac{m'}{n'}$;
11. $\forall x_1 \forall x_2 \forall x_3 \forall y(\tau_{<8,m,n,p,q>}(x_1, x_2, x_3, y) \wedge \tau_{<8,m',n',p',q'>}(x_1, x_2, x_3, y')) \to y = y'$,
    where $\frac{m}{n} = \frac{m'}{n'}$ and $\frac{p}{q} = \frac{p'}{q'}$;
12. $\forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \to \exists y(\tau_{<k,m,n,p,q,j>}(x_1, x_2, x_3, y)))$;

**(R1)**:

$$\frac{\{\forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \to \psi(x_1, x_2, x_3, y)) | \tau \text{ a fixing formula}\}}{\forall y \forall x_1 \forall x_2 \forall x_3 (coord(x_1, x_2, x_3) \wedge hp(y) \to \psi(x_1, x_2, x_3, y).}$$

**(R2)**

$$\frac{\{\forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1\leq i\leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n) \rightarrow \psi(y))|n \in \mathbf{N}, bc \text{ a Boolean combination formula}\}}{\forall y(\psi(y)).}$$

Our axiom system comprises two parts:

1. logical axioms and rules of inference;
2. non-logical axioms **(1-12)** and rules of inference **(R1)** and **(R2)** above.

On an intuitive level, assuming our standard interpretation, the meaning of the above axioms is as follows. Axioms 1 make sure that the structure is a Boolean Algebra. Axiom 2 asserts that there are at least three regions such that lines bounding them form a coordinate frame. Axiom 3 states that 0 is not convex. Axiom 4 states that if a region is a Boolean combination of half-planes, then it is convex if and only if it is a product of some of these half-planes. Axioms 5 ensures that if $\tau_i$ fix half-planes (in $ROQ(\mathbf{R}^2)$) with an empty product, then elements fixed by $\tau_i$ interpreted in *any* model of the proposed axiom system, are forced to have a product equal to 0. Axioms 6-8 together say that no half-plane can be fixed by two formulas $\tau$ and $\tau'$ with $k \neq k'$ and (in case $k \in \{5, 6, 7\}$) with $\frac{m}{n} \neq \frac{m'}{n'}$ and (when $k = 8$) $\frac{m}{n} \neq \frac{m'}{n'}$ and $\frac{p}{q} \neq \frac{p'}{q'}$. Axioms 9-11 together say that if two half-planes $a$ and $a'$ say, have fixing formulas with $k = k'$ (when $k \in \{1, 2, 3, 4\}$) or with $k = k'$ and $\frac{m}{n} = \frac{m'}{n'}$ (when $k \in \{5, 6, 7\}$) or with $k = k'$ and $\frac{m}{n} = \frac{m'}{n'}$, $\frac{p}{q} = \frac{p'}{q'}$ (when $k = 8$), then $a = a'$. Axiom schema 12 ensures that, given a coordinate system and a fixing formula, there is a half-plane fixed by this formula in reference to this coordinate system. Infinitary rule R1 states that every half-plane can be fixed in reference to a given coordinate system. Finally R2 states that every region is a Boolean combination of some half-planes. Let $\Phi$ be a set of $\mathbf{L}_{conv,\leq}$-sentences. A proof in the above axiom system is a sequence of $\mathbf{L}_{conv,\leq}$-formulas $\{\phi_\alpha\}_{\alpha<\beta}$ for some (not necessarily finite) ordinal $\beta$ such that every $\phi_\alpha$ is either an element of $\Phi$; an axiom; or the result of applying a rule of inference to some formulas $\phi_\gamma$ with $\gamma < \alpha$. If $\psi$ is the last line of such proof we write $\Phi \vdash \psi$. If $\Phi = \{\phi\}$ we write $\phi \vdash \psi$ and if $\Phi = \emptyset$ we write $\vdash \psi$ and call $\psi$ a theorem. Denote the set of theorems by $T(\mathbf{Ax})$.

**Theorem 12** (Deduction Theorem)**.** *Let $\phi$ be an $\mathbf{L}_{conv,\leq}$-sentence and $\psi$ an $\mathbf{L}_{conv,\leq}$-formula such that $\phi \vdash \psi$. Then $\vdash \phi \rightarrow \psi$.*

## 4 Soundness

In this section we prove the soundness theorem for our axiom system. The following result is standard.

**Theorem 13.** *Let $A_1, \ldots, A_n$ be sets in $\mathbf{R}^2$. If $A_1, \ldots, A_n$ are convex, then their intersection $\bigcap_{1\leq i\leq n} A_i$, if non-empty, is convex.*

**Theorem 14.** *Let $A \in ROQ(\mathbf{R}^2)$ be any convex set and let $A_1, A_2, \ldots, A_n$ be half-planes in $ROQ(\mathbf{R}^2)$. $A$ is expressible as a sum of products of $A_1, A_2, \ldots, A_n$ if and only if $A = \prod_{i\in I} A_i$, $I \subseteq \{1, \ldots, n\}$.*

*Sketch proof.* Let $a$ be a convex region expressible as a sum of products of some half-planes. Note that $a$ cannot be the empty set by the definition of convexity. The proof proceeds by eliminating all "internal" lines of $a$ (and thus decreasing the number of half-planes involved) in a way that preserves convexity. Eventually, we eliminate all internal lines and are left with a convex set equal to $a$ which is a product of the remaining "bounding" lines. □

**Theorem 15.** *The inference rules are truth-preserving.*

*Proof.* R1: Observe that given any coordinate system and a half-plane $h$, the position of $h$ in refence to this coordinate system falls into eight categories mentioned in the outline of the proof of the theorem 12. Since the intersection point of two non-parallel rational lines is a point with rational coordinates, clearly any such an arrangement is expressible by some fixing formula $\tau$. The result then follows.

R2: The result is obvious, as every $r \in ROQ(\mathbf{R}^2)$ is a rational polygon and so it is a Boolean combination of some rational half-planes. □

We are ready to state the main result of this section.

**Theorem 16** (Soundness Theorem)**.** *Let $\psi$ be an $\mathbf{L}_{conv,\leq}$-sentence. If $\psi \in T(\mathbf{Ax})$ then $\mathbf{M} \models \psi$.*

*Proof.* We are required to show that all the axioms are true in $\mathbf{M}$ and that the inference rules are truth preserving. It should be clear why Axioms 2, 5-12 are true in $\mathbf{M}$. Since $ROQ(\mathbf{R}^2)$ is a Boolean Algebra, Axioms 1 hold. Since by definition 0 is non-convex, axiom 3 holds. Axioms 4 is true by the virtue of theorem 15 and rules R1 and R2 by theorem 15 □.

## 5 Completeness

In this section we prove the completeness theorem for our axiom system. We make extensive use of the following, classical results. Let $\Sigma(\bar{x})$ be a set of formulas in a language $\mathbf{L}$ with free variables in $\bar{x}$. An $\mathbf{L}$-structure $\mathbf{A}$ is said to *realise* $\Sigma(\bar{x})$ if there exists a tuple $\bar{a}$ from $A$ satisfying every $\sigma(\bar{x}) \in \Sigma(\bar{x})$. We say that $\mathbf{A}$ *omits* $\Sigma(\bar{x})$ if $\mathbf{A}$ does not realise $\Sigma(\bar{x})$. An $\mathbf{L}$-theory $T$ is said to *locally realise* $\Sigma(\bar{x})$ if there is a formula $\phi(\bar{x})$ such that $\phi(\bar{x})$ is consistent with $T$ and for all $\sigma(\bar{x}) \in \Sigma(\bar{x})$, $T \models \forall\bar{x}\,(\phi(\bar{x}) \rightarrow \sigma(\bar{x}))$. We say that $T$ *locally omits* $\Sigma(\bar{x})$ if $T$ does not locally realise $\Sigma(\bar{x})$. In rds, $T$ locally omits $\Sigma(\bar{x})$ if for every formula $\phi(\bar{x})$ consistent with $T$ there exists $\sigma(\bar{x}) \in \Sigma(\bar{x})$ such that $T \not\models \forall\bar{x}\,(\phi(\bar{x}) \rightarrow \sigma(\bar{x}))$.

We modify these standard notions as follows.

**Definition 9.** *A theory $T$ is said to* locally realise $\Sigma(\bar{x})$ given a formula $\alpha(\bar{x})$ *if there exists $\phi(\bar{x})$ such that $\phi(\bar{x}) \wedge \alpha(\bar{x})$ is consistent with $T$ and for all $\sigma(\bar{x}) \in \Sigma(\bar{x})$, $T \models \forall\bar{x}\,(\phi(\bar{x}) \wedge \alpha(\bar{x}) \rightarrow \sigma(\bar{x}))$. Otherwise $\phi(\bar{x})$ locally omits $\Sigma(\bar{x})$ given $\alpha(\bar{x})$ in $T$.*

We have the following theorem.

**Theorem 17** (Conditional Omitting Types Theorem)**.** *Let $\Sigma_0(\bar{x})$ be a type of arity $n$ and let $\Sigma_1(\bar{y})$ be a type of arity $m$. Suppose $T$ is a consistent theory in a countable language. If $T$ (i) locally omits $\Sigma_0(\bar{x})$ given $\alpha(\bar{x})$;(ii) locally omits $\Sigma_1(\bar{y})$, then $T$ has a countable model omitting $\Sigma_1(\bar{y})$ and $\Sigma_2(\bar{x}) = \{\alpha(\bar{x}) \wedge \sigma_0(\bar{x}) : \sigma_0(\bar{x}) \in \Sigma_0(\bar{x})\}$.*

*Sketch proof.* The proof of this theorem is a slight modification of proofs of the classical omitting types theorem and extended omitting types theorem (Cf. [3]). □

We are now ready to state the main theorem of this section:

**Theorem 18** (Completeness Theorem)**.** *Let $\psi$ be an $\mathbf{L}_{\leq,conv}$-sentence. If $\mathbf{M} \models \psi$ then $\psi \in T(\mathbf{Ax})$.*

*Proof.* Let $\psi$ be an $\mathbf{L}_{\leq,conv}$-sentence such that $\psi \notin T(\mathbf{Ax})$. We are required to show that $\mathbf{M} \not\models \psi$. Now let

$$T = \{\phi : \neg\psi \vdash \phi\}.$$

By the deduction theorem $T$ is consistent. Consider the following sets of formulas:

1. $\Sigma_1(x_1, x_2, x_3, y) = \{coord(x_1, x_2, x_3) \wedge hp(y) \wedge \neg\tau(x_1, x_2, x_3, y) : \tau$ a fixing formula$\}$,

2. $\Sigma_2(x) = \{\neg\exists y_1 \ldots \neg\exists y_n(\bigwedge_{1\leq i\leq n} hp(y_i) \wedge x = bc(y_1, \ldots, y_n)) : n \in \mathbf{N}, bc$ a Boolean combination formula$\}$.

Suppose $\Theta(x_1, x_2, x_3, y)$ is a formula such that $\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y)$ is consistent with $T$. We then have $T \not\models \forall x\forall x_1\forall x_2\forall x_3\neg(\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y))$ and

$$T \not\models \forall x_1\forall x_2\forall x_3\forall y(coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \neg\Theta(x_1, x_2, x_3, y)),$$

so by R1:

$T \not\models \forall x\forall x_1\forall x_2\forall x_3((coord(x_1, x_2, x_3) \wedge hp(y) \wedge \tau(x_1, x_2, x_3, y)) \rightarrow \neg\Theta(x_1, x_2, x_3, y))$
for some $\tau$.

Hence $\Theta(x_1, x_2, x_3, y) \wedge coord(x_1, x_2, x_3) \wedge hp(y)$ consistent with $T$ implies

$$T \not\models \forall x_1\forall x_2\forall x_3\forall y(\Theta(x_1, x_2, x_3, y) \wedge (coord(x_1, x_2, x_3) \wedge hp(y) \rightarrow \neg\tau(x_1, x_2, x_3, y)),$$

for some $\tau$.

In other words, $T$ locally omits $\Sigma(x_1, x_2, x_3, y) = \{\neg\tau(x_1, x_2, x_3, y) : \tau$ a fixing formula $\}$ given $coord(x_1, x_2, x_3) \wedge hp(y)$.

Now suppose $\Theta(x)$ is any formula consistent with $T$. We then have

$$T \not\models \forall x\neg\Theta(x)$$

and by R2: $T \not\models \forall y(\exists x_1 \ldots \exists x_n(\bigwedge_{1\leq i\leq n} hp(x_i) \wedge y = bc(x_1, \ldots, x_n)) \rightarrow \neg\Theta(y))$ for some $n \in \mathbf{N}$ and some $bc$, so $\Theta(x)$ consistent with $T$ implies

$$T \not\models \forall x(\Theta(x) \rightarrow \neg(\exists y_1 \ldots \exists y_n(\bigwedge_{1\leq i\leq n} hp(y_i) \wedge bc(x, y_1, \ldots, y_n))$$

for some $n \in \mathbf{N}$ and some $bc$. In other words, $T$ locally omits $\Sigma_2(x)$.

By the conditional omitting types theorem there exists a countable model $\mathbf{A}$ of $T$ omitting $\Sigma_1(x_1, x_2, x_3, y)$ and $\Sigma_2(x)$ .

A more intuitive way of saying that $\mathbf{A}$ omits $\Sigma_1$ amd $\Sigma_2$ is that for every element $a$ of $A$ and any $l_1, l_2, l_3 \in A$ forming a coordinate frame, $a$ can be expressed as a Boolean combination of some $b_1, \ldots, b_k \in A$ such that $\mathbf{A} \models \bigwedge_{1\leq i\leq k} hp[b_i]$ and $\mathbf{A} \models \bigwedge_{1\leq i\leq k} \tau_i[l_1, l_2, l_3, b_i]$, where $\tau_i$ is a fixing formula for $b_i$.

Since the carrier set of $\mathbf{A}$ is countable we can enumerate its elements :

$$A = \{a_1, a_2, a_3, \ldots\}.$$

We fix this notation for the reminder of this section. Assume WLOG that $|A| > 2$ and that, by axiom 2, $a_1, a_2, a_3$ are such that

$$\mathbf{A} \models coord[a_1, a_2, a_3].$$

By the fact that $\mathbf{A}$ omits $\Sigma_2$, for each $a \in A$ we have that

$$\mathbf{A} \models a = bc(b_1, \ldots, b_n)$$

for some $b_1, \ldots, b_n \in A$ such that

$$\mathbf{A} \models \bigwedge_{1\leq i\leq n} hp[b_i],$$

Since $\mathbf{A}$ omits $\Sigma_1$, for each $i \in \{1, \ldots, n\}$ there exists a fixing formula $\tau$ such that $\mathbf{A} \models \tau[a_1, a_2, a_3, b_i]$.

We now proceed to define a mapping $e : A \rightarrow ROQ(\mathbf{R}^2)$.

We start by defining a mapping $e^{(k)}$ for each initial segment $a_1, \ldots, a_k$ of elements of $A$. By the above considerations let $a_i = bc(b_1^{(i)}, \ldots, b_{N(i)}^{(i)})$ for each $i \in \{1, \ldots, k\}$. Fix $h_1, h_2, h_3 \in ROQ(\mathbf{R}^2)$ such that

$$\mathbf{M} \models coord[h_1, h_2, h_3]$$

and define $e^{(k)}(a_i) = h_i, i \in \{1, 2, 3\}$.

Note that by theorem 12 there exists a unique half-plane $h_j^{(i)} \in ROQ(\mathbf{R}^2)$ such that

$$\mathbf{M} \models \tau_j^{(i)}[h_1, h_2, h_3, h_j^{(i)}].$$

We define $e^{(k)}(b_j^{(i)}) = h_j^{(i)}$.

**Lemma 1.** *Let $a_1, \ldots, a_k$ be some initial segment of $A$. Then the mapping $e^{(k)}$ is well defined.*

*Proof.* Firstly note that for each half-plane $b_i$ involved in a construction of any of $a_1, \ldots, a_k$ it follows from axioms 6-8 that if $\mathbf{A} \models \tau_{<k,m,n,p,q>}[a_1, a_2, a_3, b_i]$ and $\mathbf{A} \models \tau'_{<k',m',n',p',q'>}[a_1, a_2, a_3, b_i]$ then $k = k'$ and furthermore (i) if $k \in \{5, 6, 7\}$ then $\frac{m}{n} = \frac{m'}{n'}$; (ii) if $k = 8$ then $\frac{m}{n} = \frac{m'}{n'}$ and $\frac{p}{q} = \frac{p'}{q'}$.

Now let $b$ and $b'$ be two half-planes involved in a contstruction of some $a$ and $a'$ respectively, such that $\mathbf{A} \models \tau[a_1, a_2, a_3, b]$, $\mathbf{A} \models \tau'[a_1, a_2, a_3, b']$ and $b = b'$. By the definition of $e^{(k)}$, $b$ and $b'$ are mapped to some $h$ and $h'$, respectively. We are required to show that $h = h'$. But this follows from the fact that $\mathbf{A} \models \tau'[a_1, a_2, a_3, b]$ (and so the respective conditions, as described above, are satisfied) and theorem 12. □

**Lemma 2.** *Let $a_1, \ldots, a_k$ be some initial segment of $A$. Then the mapping $e^{(k)}$ is injective.*

*Proof.* Let $b$ and $b'$ be two halfplanes involved in a construction of any of $a_1, \ldots, a_k$. We need to show that if $e^{(k)}(b) = h$ and $e^{(k)}(b') = h'$ are such that $h = h'$ then $b = b'$. Let $\mathbf{M} \models \tau_{<k,m,n,p,q,j>}[h_1, h_2, h_3, h]$ and

$\mathbf{M} \models \tau'_{<k',m',n',p',q',j'>}[h_1, h_2, h_3, h']$. By theorem 12 it follows that for these fixing formulas $k = k'$ and $j = j'$ furthermore if $k \in \{5, 6, 7\}$ then $\frac{m}{n} = \frac{m'}{n'}$ and if $k = 8$ then $\frac{m}{n} = \frac{m'}{n'}$ and $\frac{p}{q} = \frac{p'}{q'}$. The result then follows from axioms 9-11. $\square$

Now, since $A$ is a Boolean Algebra, $a_1, \ldots, a_k$ generate a subalgebra of $A$. We can extend $e^{(k)}$ homomorphically in the obvious way.

**Lemma 3.** *For any initial segment $a_1, \ldots, a_k \in A$ the mapping $e^{(k)}$ is Boolean algebra isomorphism.*

*Proof.* It follows from axiom 5 that $e^{(k)}$ is a monomorphism. It is onto by definition. $\square$

**Lemma 4.** *For any initial segment $a_1, \ldots, a_k \in A$ the mapping $e^{(k)}$ is an embedding.*

*Proof.* By lemma 3 $e^{(k)}$ is a Boolean Algebra isomorphism. We are required to show that the following holds:

$\mathbf{A} \models conv[a_i]$ if and only if $\mathbf{M} \models conv[e^{(k)}(a_i)]$ where $1 \leq i \leq k$;

Suppose $\mathbf{A} \models conv[a_i]$, by axiom 3 $a_i \neq 0$ and by axioms 4 we have $\mathbf{A} \models a_i = \prod_{j \in I} b_j$, for some $I \subseteq \{1, \ldots, N(i)\}$. Therefore $e^{(k)}(a_i) \neq 0$ (lemma 3) and by definition

$$e^{(k)}(a_i) = \prod_{j \in I} e^{(k)}(b_j).$$

Since $e^{(k)}(b_j)$ are half-planes (and as such convex) $\mathbf{M} \models conv[e^{(k)}(a_i)]$.

Conversely, suppose $\mathbf{M} \models conv[e^{(k)}(a_i)]$, since each $a_i$ is a Boolean combination of some $b_1, \ldots, b_{N(i)}$ we have

$$\mathbf{A} \models conv[bc(b_1, \ldots, b_{N(i)})].$$

Then (by theorem 14) $e^{(k)}(a_i) = \prod_{1 \leq j \leq N(i)} e^{(k)}(b_j)$ for some selection of half-planes $e(b_1), \ldots, e^{(k)}(b_{N(i)})$. Therefore, since $e^{(k)}$ is a Boolean algebra homomorphism

$$e^{(k)}(a_i) = e^{(k)}(\prod_{1 \leq j \leq N(i)} b_j)$$

and since it is injective $a_i = \prod_{1 \leq j \leq N(i)} b_j$, $a_i \neq 0$ with

$$\mathbf{A} \models \bigwedge_{1 \leq j \leq N(i)} hp(b_j)$$

Hence, by axioms 3 and 4, we have $\mathbf{A} \models conv[a_i]$. $\square$

**Lemma 5.** *Let $e : A \to ROQ(\mathbf{R}^2)$ be defined as $e = \bigcup_{i=1}^{\infty} e^{(i)}$. Then $e$ is an embedding.*

*Proof.* We need to show that $e$ is injective.But this is obvious in a view of lemma 4. $\square$

**Lemma 6.** *The mapping $e : A \to ROQ(\mathbf{R}^2)$ is an isomorphism.*

*Proof.* By axiom 12 for any selection of $k, m, n, p, q$ we can find an element $a \in A$ such that $\mathbf{A} \models \tau_{<k,m,n,p,q>}[a_1, a_2, a_3, a]$ and so $e$ is onto. $\square$

It follows that $\mathbf{A}$ is isomorphic to $\mathbf{M}$ and so $\mathbf{A} \models \phi$ if and only if $\mathbf{M} \models \phi$ for all $\mathbf{L}_{conv,\leq}$-sentences $\phi$.

Recall the way $\mathbf{A}$ is constructed. It follows that $\mathbf{A} \models \neg\psi$ but then also $\mathbf{M} \models \neg\psi$ and so $\mathbf{M} \not\models \psi$, which concludes the completeness proof. $\square$

## 6 Conclusions

We proposed an axiomatization of the theory of $\mathbf{M} = \langle conv, \leq \rangle$ with geometrical interpretation. We proved soundness and completeness for this axiomatization by constructing an abstract model $\mathbf{A}$ with certain desired properties. We used slightly modified omitting types theorem to show that $\mathbf{A}$ is countably infinite. Also, by the existence of fixing formulas in $\mathbf{M}$, allowing fixing of any rational line in reference to a given coordinate system, we showed that $\mathbf{A}$ is isomorphic to $\mathbf{M}$ which is a final step in the completeness proof.

A number of interesting research possibilities suggest themselves at this point. Can we axiomatise convexity logics where variables are set to range over different selection of the subsets of the real plane? It seems possible for the real open *algebraic* polygons and very hard for the real open polygons or indeed just regular open sets. Can we axiomatise convexity logics with variables ranging over subsets of Euclidean space of higher dimensions (in particular of the third dimension)? This seems possible in case of regular open rational/algebraic polygons. However, different methods are required for other mentioned possibilites and it is not straightforward how to do it. From the practical point of view it would be interesting to look at the propositional fragment of our convexity logic. Can we axiomatise this fragment? As things stand now, current axiomatisation is not fit for this purpose as it contains formulae with (nested) existential quantifiers.

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