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Higher-Order Coalition Logic

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Abstract. We introduce and study higher-order coalition logic, a multi modal monadic second-order logic with operators $[\{x\}\psi]\varphi$ expressing that the coalition of all agents satisfying $\psi(x)$ can achieve a state in which φ holds. We use neighborhood semantics to model extensive games of perfect information with simultaneous actions and we provide a framework reasoning about agents in the same way as it is reasoning about their abilities. We illustrate higher-order coalition logic to represent and reason about coalition formation and cooperation, we show a more general and expressive way to quantify over coalitions than quantified coalition logic, we give an axiomatization and prove completeness.

1 Introduction

Multiagent modal logic typically indexes modal operators by (sets of) agents, such that $K_1 p$ stands for agent 1 knowing p, or $C_{1,2}q$ stands for common knowledge among agent 1 and 2 that q. In logics of ability, coalition logic [18] extends classical propositional logic with operators $[C]\varphi$ for groups of agents C, read as: "the coalition C can make a choice such that φ holds" or "φ is a possible outcome for coalition C", and therefore can specify and reason about abilities of coalitions in games. Alternate-time temporal logic [3] may be seen as an extension of coalition logic as well as of computational tree logic [10] with operators $\langle\langle C\rangle\rangle\varphi$, read as: "the coalition C has a winning strategy such that φ holds", and can reason on strategies or *sequences* of choices. Moreover, these two logics have been further extended with, for example, explicit representation of actions [6], epistemic operators [22], preferences structures [9], explicit representation of strategies [23], and more [2, 8, 7, 19].

A general question in multiagent modal logic is whether the agents can be described by formulas, for example describing the roles the agents are playing, and logics of ability have been extended with quantification over coalitions [1]. In particular, Agotnes at al. [1] came up with the idea of using restrictions on the coalitions that appear in the modality of CL.

In this paper we address the following research question: *How to extend the logic presented by Ågotnes et al. [1] with a monadic second-order language to unify the language to specify coalitions with the language to talk about outcomes?*

This breaks down in the following sub-questions:

- What is the semantics and what properties can we express in this extended language?
- How to show axiomatization, soundness and completeness for such a logic?

Replying to the above questions is pivotal for the definition of a general, expressive and formal framework to represent knowledge about coalitional games.

In contrast to most work in modal agent logic, we start from first-order logic rather than propositional logic. More precisely, we start from an extension of first-order logic in which we can quantify over the subsets of the domain, called monadic second-order logic (MSO). Higher-order coalition logic is a modal extension of monadic second-order logic, in which we follow coalition logic by employing neighborhood semantics to model extensive games of perfect information with simultaneous actions [16].

Our completeness result extends a completeness proof for first-order modal logic with neighborhood semantics [4] by handling set variables.

In this paper we do not consider decidable fragments of higher-order coalition logic. Jamroga and Seylan in [15] introduce a decidable extension of coalition description logic expressing state of individual agents in a limited way. Also the integration of some form of quantification over coalitions by extending coalition description logics is left for further research.

The paper is structured as follows. First we repeat quantified coalition logic, then we introduce higher-order coalition logic, and we analyze how it extends the expressive power of quantified coalition logic. Finally we provide an axiomatization, prove completeness, and show a translation from quantified coalition logic to higher-order coalition logic.

2 Quantified Coalition Logic

This section is based on [18] and [1], to which we refer for a detailed discussion.

Coalition Logic (CL) is a propositional modal logic, with modalities indexed by a *coalition*, i.e., a subset of a given finite and bounded set of agents Ag. In CL we write $[C]\varphi$, where $[C]$ is a modality and φ a formula, to intuitively express that 'C *can achieve* φ', or, that 'C *is effective* for φ', or that 'C *has a choice* such that φ'.

Formulae of CL are defined by the following grammar:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid [C]\varphi$$

where p ranges over the set of Boolean variables Φ_0 and C is a subset of a fixed set of agents Ag.

Definition 1 (Model in CL) *A model $\mathcal{M}$ for CL is a triple $\mathcal{M} = \langle S, N, \pi\rangle$ where*

- $S = \{s_1, \ldots, s_n\}$ *is a finite non-empty set of* states.
- $N : \mathcal{P}(Ag) \times S \to \mathcal{P}(\mathcal{P}(S))$ *is a* neighborhood function[5] *where if $T \in N(C, s)$ then, from state s, the coalition C can cooperate to ensure that the next state will be a member of T.*

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[5] Called also *effectivity function*.

- $\pi : S \rightarrow \mathcal{P}(\Phi_0)$ *is a valuation function, which for every state* $s \in S$ *gives the set* $\pi(s)$ *of Boolean variables that are satisfied at* s.

An *interpretation* for CL is a pair $\mathcal{M}, s$ where $\mathcal{M}$ is a model and s is a state in $\mathcal{M}$. The satisfaction relation "$\models_{CL}$" for CL holds between interpretations and formulae of CL. We say that coalition C *can enforce* φ in s if for some $T \in N(C, s)$, φ is true in all $t \in T$. That is, C can make a choice such that, irrespective of the others' choices, φ will hold. The satisfaction relation is defined as usual for atomic variables and $\top, \neg, \vee$, where for the modal case we have:

- $\mathcal{M}, s \models_{CL} [C]\varphi$ iff $\exists T \in N(C, s)$ such that $\forall t \in T$, we have $\mathcal{M}, t \models_{CL} \varphi$

As reported in [17], it is possible to define a number of constraints on neighborhood functions, depending upon exactly which kind of scenario they are intended to model. The most general structures to deal with extensive games with simultaneous moves are *weak playability models*, a model $\mathcal{M}$ is weakly playable iff the neighborhood function has the properties reported in Def. 2

Definition 2 *A neighborhood function N is* weakly playable *if it satisfies the following properties:*

1. Outcome monotonicity*:* $\forall X, X', C : [X \subseteq X' \subseteq S, C \subseteq Ag]$ *if* $X \in N(C, s)$ *then* $X' \in N(C, s)$ *(i.e., if a coalition C can achieve an ouctcome in a set, it can achieve an outcome in any bigger set).*
2. $\emptyset \notin N(Ag, s)$ *(i.e., the grand coalition Ag cannot achieve $\bot$).*
3. $\forall C', C : [C' \subseteq C \subseteq Ag]$ *if* $\emptyset \in N(C, s)$ *then* $\emptyset \in N(C', s)$ *(i.e., together with the second, fourth and fifth assumptions,* no *coalition can enforce $\bot$).*
4. $\forall C \subseteq Ag$ *if* $\emptyset \notin N(\emptyset, s)$ *then* $S \in N(C, s)$*(i.e., together with the first assumption we have that any coalition C can achieve* something*).*
5. Superadditivity*:* $\forall X_1, X_2, C_1, C_2 : [X_1, X_2 \subseteq S; C_1, C_2 \subseteq Ag; (C_1 \cap C_2 = \emptyset)]$ *If* $X_1 \in N(C_1, s)$ *and* $X_2 \in N(C_2, s)$ *then* $X_1 \cap X_2 \in N(C_1 \cup C_2, s)$ *(i.e., if C_1 can enforce X_1 and C_2 can achieve X_2, they can both exercise their ability in order to enforce $X_1 \cap X_2$).*
6. Ag-maximality*:* $\forall X : [X \subseteq S]$ *If* $(S \setminus X) \notin N(\emptyset, s)$ *then* $X \in N(Ag, s)$ *(i.e., by contraposition, if Ag cannot enforce X, then $S \setminus X$ is already enforced).*

As reported in [1], if we have n agents in Ag, and one wants to express that *some* coalition can enforce some atomic property p, one needs to enumerate 2^n disjunctions of the form $[C]p$. The idea behind Quantified Coalition Logic (QCL) is to avoid this blow-up in the length of formulae. Informally, QCL is a propositional modal logic, containing an indexed collection of unary modal operators $\langle P \rangle \varphi$ and $[P]\varphi$. The intended interpretation of $\langle P \rangle \varphi$ is that *there exists a set of agents C, satisfying predicate P, such that C can achieve φ*. We refer to expressions P as *coalition predicates*, in the following we report the language of coalition predicates.

The syntax of the coalition predicates is given by the following grammar:

$$P ::= subseteq(C) \mid supseteq(C) \mid \neg P \mid P \vee P$$

where $C \subseteq Ag$ is a set of agents and $subseteq$ and $supseteq$ are two atomic predicates. The circumstances under which a concrete coalition Co satisfies a coalition predicate P, are specified by a satisfaction relation "$\models_{cp}$", defined as follows[6]:

- $Co \models_{cp} subseteq(C)$ iff $Co \subseteq C$
- $Co \models_{cp} supseteq(C)$ iff $Co \supseteq C$

The Quantified Coalition Logic (QCL) presented in [1] extends Pauly's Coalition Logic by introducing the above mentioned coalition predicates. Formulae of QCL are defined by the following grammar (with respect to a set Φ_0 of Boolean variables, a fixed set Ag of agents, and the language of coalition predicates):

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle P \rangle \varphi \mid [P]\varphi$$

where $p \in \Phi_0$ is an atomic proposition and P is a coalition predicate over Ag[7].

The satisfaction relation "$\models_{QCL}$" for QCL holds between interpretations and formulae of QCL, and it is defined for the modal operators as follows:

- $\mathcal{M}, s \models_{QCL} \langle P \rangle \varphi$ iff $\exists C \subseteq Ag : C \models_{cp} P$ and $\exists S \in N(C, s)$ such that $\forall s' \in S$, we have $\mathcal{M}, s' \models_{QCL} \varphi$
- $\mathcal{M}, s \models_{QCL} [P]\varphi$ iff $\forall C \subseteq Ag : C \models_{cp} P$ implies $\exists S \in N(C, s)$ such that $\forall s' \in S$, we have $\mathcal{M}, s' \models_{QCL} \varphi$

In [1] it is proved that QCL is exponentially more succinct than CL i.e., if we want to translate a QCL formula of the type $\langle subseteq(Ag) \rangle \varphi$ into a CL we get a formula which is exponentially longer, since it has to explicitly enumerate all coalitions in Ag.

3 Higher-Order Coalition Logic

In the following we present Higher-Order Coalition Logic (HCL) which extends Coalition Logic by using a monadic second-order language (MSO). MSO is the extension of first-order logic that allows quantification over the subsets of the domain. Predicates of any arity may appear in monadic formulae, but they may not be quantified over. Monadic theories are useful in various branches of mathematical logic and its applications [13]. Often they have a reasonable level of expressiveness sufficient to formalize interesting features but simple enough to be manageable. For deeper treatment of monadic theories, we refer to [5], whereas for a description of second-order languages we refer to [11].

We assume the usual notions of variables, predicates, connectives $\wedge, \vee, \rightarrow, \neg$, quantifiers $\forall, \exists$ and the notions of free and bound variables. Let $\mathcal{V} = \mathcal{V}_I \cup \mathcal{V}_S$ be a countable collection of variables where $\mathcal{V}_I$ and $\mathcal{V}_S$ are partitions for *individual* and *set* variables respectively. For each natural number $n \geq 1$, there is a (countable) set of n-place predicate symbols $(F, G, \ldots)$. Formulas of HCL are defined by the following grammar:

$$\varphi ::= F(x_1, \ldots, x_n) \mid Xx \mid (\neg\varphi) \mid (\varphi \vee \varphi) \mid \forall X(\varphi) \mid \forall x(\varphi) \mid [\{x\}\varphi]\varphi \mid \langle \{x\}\varphi \rangle \varphi$$

where

- $F(x_1, \ldots x_n)$ is a first-order atomic formula [8].
- x is a first-order variable.
- X is a set variable.
- $\{x\}\psi$ is a group operator which, intuitively represents the set of all elements d such that $\psi[d/x]$ holds.

We use $\exists X\varphi$ (resp. $\exists x\varphi$) as a shorthand for $\neg\forall X\neg\varphi$ (resp. $\neg\forall x\neg\varphi$).

[6] Satisfaction for $\neg$ and $\vee$ is as usual.
[7] Notice that in $[P]\varphi$ the language of P and φ are disjoint.
[8] In general we write $\varphi(x)$ when x is the only free variable in φ.

Definition 3 *A second-order **constant domain coalition frame** is a tuple $\langle W, N, D\rangle$ where:*

- *D is any non-empty set of agents called the **domain**.*
- *W is a set of states.*
- *$N : W \times \mathcal{P}(D) \to \mathcal{P}(\mathcal{P}(W))$ is a neighborhood function.*

We now restrict our study to a specific class of frames, which are the higher-order counter part of weakly playable frames of [17].

Definition 4 *A constant domain frame $\mathcal{F} = \langle W, N, D\rangle$ is weakly playable iff the neighborhood function satisfies the properties in Def 2.*

In this work we concentrate over weakly playable frames because they are the most general structures to cope with extensive games with simultaneous moves.

Definition 5 *A second-order **constant domain coalition model** based on a frame $\mathcal{F} = \langle W, N, D\rangle$ is a a tuple $\mathcal{M} = \langle W, N, D, I, \sigma\rangle$ together with a set $\mathcal{A}_s$, where*

- *I is a classical first-order interpretation function where for each n-ary predicate symbol F, $I(F, w) \subseteq D^n$.*
- *σ is a function which assigns objects to individual variables and which is parametrized by w for set variables such that $\sigma(X, w) \subseteq D$[9]. We let $\sigma[u/x]$ be an assignment which differs from σ only in assigning u to x (similarly for $\sigma[U/X]$ where $\sigma[U/X](X, w) = U$ for every w).*
- *$\mathcal{A}_s \subseteq \mathcal{P}(\mathcal{P}(D))$ is a family of* admissible coalitions *such that $U \in \mathcal{A}_s$ iff there exists a well-formed formula φ such that $U = \{a \in D \mid \mathcal{M} \models_{\sigma[a/x]} \varphi(x)\}$. Intuitively, we consider admissible all and only the coalitions that are definable by an HCL formula.*

Let $\mathcal{M} = \langle W, N, D, I, \sigma\rangle$ be any constant domain coalition model, we define the HCL satisfaction relation $\models$ as follows:

- $\mathcal{M}, w \models_\sigma F(x_1, \ldots, x_n)$ iff $\langle \sigma(x_1), \ldots, \sigma(x_n)\rangle \in I(F, w)$ for each n-place predicate symbol F
- $\mathcal{M}, w \models_\sigma \neg\varphi$ iff $\mathcal{M}, w \not\models_\sigma \varphi$
- $\mathcal{M}, w \models_\sigma \varphi \vee \psi$ iff $\mathcal{M}, w \models_\sigma \varphi$ or $\mathcal{M}, w \models_\sigma \psi$
- $\mathcal{M}, w \models_\sigma Xx$ iff $\sigma(x) \in \sigma(X, w)$
- $\mathcal{M}, w \models_\sigma \forall x\varphi$ iff for every object $d \in D$ such that for $\sigma' = \sigma[d/x]$ we have $\mathcal{M}, w \models_{\sigma'} \varphi$
- $\mathcal{M}, w \models_\sigma \forall X\varphi$ iff for every set $U \in \mathcal{A}_s$, we have $\mathcal{M}, w \models_{\sigma[U/X]} \varphi$
- $\mathcal{M}, w \models_\sigma [\{x\}\psi]\varphi$ iff $(\varphi)^{\mathcal{M},\sigma} \in N(w, U)$ where $U = \{d \mid \mathcal{M}, w \models_\sigma \psi[d/x]\}$
- $\mathcal{M}, w \models_\sigma \langle\{x\}\psi\rangle\varphi$ iff $W - (\varphi)^{\mathcal{M},\sigma} \notin N(w, U)$ where $U = \{d \mid \mathcal{M}, w \models_\sigma \psi[d/x]\}$

Where $(\varphi)^{\mathcal{M},\sigma} \subseteq W$ is the set of states $w \in W$ such that $\mathcal{M}, w \models_\sigma \varphi$. In the definition of truth of modal formulas we refer to U as *the coalition identified by* ψ. Notice that the second-order quantification ranges over a family of admissible subsets (i.e., $\mathcal{A}_s$), this semantics is often referred to as *general* (or Henkin).

4 HCL as a Representation Language

Before diving into axiomatization and completeness of the logic we want to stress the added value of a highly expressive language like HCL.

As we underlined in the introduction, both CL and QCL have the strong limitation that the language to express coalitions and the language to talk about outcomes are separate, moreover the adopted propositional language puts serious limits on the practical employment of such logics to represent knowledge about games in a compact and effective way.

HCL contributes to extend CL expressiveness in three ways:

1. Being a first-order language, agents are elements of the domain and can have properties and relationships among themselves. For instance, we can assign roles to agents and organize them in hierarchies, the formula
$$\forall x(super\_user(x) \to user(x))$$
means that every agent that is a $super\_user$ is also a $user$.
2. Set variables permit to quantify over coalitions in a more general way than QCL. For instance the formula
$$\forall X(\forall x(Xx \to user(x)) \to [\{y\}Xy]\varphi)$$
expresses that *every* coalition such that all of its members are $users$ can achieve φ. Notice that by exploiting the fact that every $super\_user$ is a $user$ we get that
$$[\{x\}super\_user(x)]\varphi$$
The language of QCL coalition predicates does not take into account any relational property between agents.
3. In $[\{x\}\varphi(x)]\psi$ the formula φ describing the coalition is in the same language of ψ. This feature lets us express complex properties on the basis of the relationships between agents in different coalitions, for instance:
$$[\{x\}\varphi(x)]\psi \to [\{y\}\exists x(\varphi(x) \wedge collaborates(y, x))]\psi$$
means that if the coalition represented by $\varphi(x)$ can achieve ψ then also the group of all agents which collaborates with at least one member of $\varphi(x)$ can. With HCL is then possible to express *conditional* winning strategies by relying on the attributes of the agents composing a coalition.

Because HCL is a higher-order language, the following question arises:

What is the relationship between quantifiers and modal operators?

In particular, could it be sensible to have the following formula as a theorem of HCL?

$$[\{x\}\varphi(x)]\forall y\psi(y) \to \forall y[\{x\}\varphi(x)]\psi(y)$$

The above formula is a multi-modal variation of the *Converse Barcan Formula* (CBF). We argue that the reply to the above question should be affirmative. To convince yourself, think about the following example:

$$[\{x\}x = Valerio](\forall y(phd\_supervisor(y) \to happy(y))) \to$$
$$\forall y([\{x\}x = Valerio](phd\_supervisor(y) \to happy(y)))$$

In fact, we can read it as:"If Valerio can achieve a state in which all of his supervisors are happy then, for every single supervisor Valerio can achieve a state in which that supervisor is happy."

What does not seem to be sensible for a coalition logic is to have the *Barcan Formula* (BF) as a theorem, i.e.

$$\forall y[\{x\}\varphi(x)]\psi(y) \to [\{x\}\varphi(x)]\forall y\psi(y)$$

In fact the corresponding example

[9] With this definition of σ we make set variables assignment dependent on states, this is due because set variables can be substituted with monadic predicates which have an interpretation that *depends* on states too.

$$\forall y([\{x\}x = Valerio](phd\_supervisor(y) \to happy(y))) \to [\{x\}x = Valerio](\forall y(phd\_supervisor(y) \to happy(y)))$$

does not necessarily be true, the fact that Valerio, given a supervisor, can make him happy does not mean that he can achieve a state in which all of them simultaneously are.

The intuitive need of having CBF valid and not BF when reasoning about coalitions makes impossible the use of *any* relational (e.g., Kripke) semantics for higher-order coalition logic with constant domain, in fact it is well known that *the Barcan formula is valid in all first-order relational models with constant domains* [14].

Surprisingly, from theory of neighborhood semantics we know that CBF is valid for the class of all constant domain *supplemented*[10] frames but not BF [4].

Theorem 1 *In HCL for any formula ψ, φ and variable $\mathcal{X} \in \{x, X\}$*

- $\not\models \forall \mathcal{X}[\{x\}\psi]\varphi(\mathcal{X}) \to [\{x\}\psi]\forall \mathcal{X}\varphi(\mathcal{X})$
- $\models [\{x\}\psi]\forall \mathcal{X}\varphi(\mathcal{X}) \to \forall \mathcal{X}[\{x\}\psi]\varphi(\mathcal{X})$

Proof. See [4] (Observation 3.13). □

5 The Axiomatization of HCL

The axiomatization of HCL consists of three parts: the standard axiomatization of first-order logic (figure 1) and monadic second-order logic (figure 2) together with the specific axioms of HCL (figure 3). The MSO axiomatization mimics the axiomatization of first-order logic (without equality), the only difference is in the $COMP$ axiom[11] which characterizes *general* semantics for monadic second-order logic. The similarity in the axiomatizations of Fig. 1 and 2 comes from the fact the MSO under general semantics can be re-interpreted over multi-sorted first-order logic, for a deeper treatment of completeness of MSO under general semantics we refer to [11] (Section 4.4). In figure 3, we have the specific axioms of HCL. Axiom E establishes the duality of $[\{x\}\psi]$ in terms of $\langle\{x\}\psi\rangle$, while axioms $M, Ag\bot, \top, \bot, N, S$ are the direct translation of the six conditions in Definition 2.

FO1	Tautologies of sentential calculus
FO2	$\vdash \forall x\varphi \to \varphi[t/x]$, where t is substitutable for x in φ
FO3	$\vdash \forall x(\varphi \to \psi) \to (\forall x\varphi \to \forall x\psi)$
FO4	$\vdash \varphi \to \forall x\varphi$, where x does not occur free in φ
FO5	$\vdash x = x$
FO6	$\vdash x = y \to (\varphi \to \psi)$ where φ is atomic and ψ is obtained from φ by replacing x in zero or more (but not necessarily all) places by y.
MP	If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$
1GEN	If $\vdash \varphi$, then $\vdash \forall x\varphi$

Figure 1. Axioms and rules of first-order logic

COMP	$\vdash \exists X\forall x(Xx \leftrightarrow \varphi)$, where X does not occur free in φ
MSO1	$\vdash \forall X\varphi \to \varphi[X/T]$, where T (which is either a set variable or a monadic predicate) is substitutable in φ for X.
MSO2	$\vdash \forall X(\varphi \to \psi) \to (\forall X\varphi \to \forall X\psi)$
MSO3	$\vdash \varphi \to \forall X\varphi$, where X does not occur free in φ
2GEN	if $\vdash \varphi$, then $\vdash \forall X\varphi$

Figure 2. Axioms and rules of monadic second-order logic

E	$\vdash [\{x\}\psi]\varphi \leftrightarrow \neg\langle\{x\}\psi\rangle\neg\varphi$
M	$\vdash [\{x\}\psi](\varphi_1 \wedge \varphi_2) \to ([\{x\}\psi]\varphi_1 \wedge [\{x\}\psi]\varphi_2)$
$Ag\bot$	$\vdash \neg[\{x\}\top]\bot$
$\top$	$\vdash \neg[\{x\}\bot]\bot \to [\{x\}\psi]\bot$
$\bot$	$\vdash (\psi_1 \to \psi_2) \to ([\{x\}\psi_2]\bot \to [\{x\}\psi_1]\bot)$
N	$\vdash \neg[\{x\}\bot]\neg\varphi \to [\{x\}\top]\varphi$
S	$\vdash (\neg\exists x(\psi_1 \wedge \psi_2)) \wedge ([\{x\}\psi_1]\varphi_1 \wedge [\{x\}\psi_2]\varphi_2) \to [\{x\}(\psi_1 \vee \psi_2)](\varphi_1 \wedge \varphi_2)$
RE	$\vdash (\varphi_1 \leftrightarrow \varphi_2) \to ([\{x\}\psi]\varphi_1 \leftrightarrow [\{x\}\psi]\varphi_2)$
$NORM$	$\vdash \forall x(\psi_1 \leftrightarrow \psi_2) \to [\{x\}\psi_1]\varphi \leftrightarrow [\{x\}\psi_2]\varphi$

Figure 3. Specific Axioms and rules of HCL

6 Completeness

In this section we discuss the completeness of HCL. First of all we need some definitions, let Γ be a set of HCL formulas.

Definition 6 *A set Γ has the $\forall-$**property** iff for each formula $\varphi \in \Gamma$:*

- *for each individual variable x, there is some variable y, called the witness, such that $\varphi[y/x] \to \forall x\varphi(x) \in \Gamma$.*
- *for each set variable X, there is some set variable Y, called the set witness, such that $\varphi[Y/X] \to \forall X\varphi(X) \in \Gamma$.*

The proof of the following Lemma can be found in [14]

Lemma 1 *If Γ is a consistent set of formulas of $\mathcal{L}_1$, then there is a consistent set of formulas Δ of $\mathcal{L}_1^+$ with the $\forall$-property such that $\Gamma \subseteq \Delta$, where $\mathcal{L}_1^+$ is the language $\mathcal{L}_1$ with countably many new variables and monadic predicates.*

We can now define the canonical model for HCL. Given an HCL language $\mathcal{L}_1$, let $\mathcal{L}_1^+$ be the extension of $\mathcal{L}_1$ used in Lemma 1 and $\mathcal{V}^+ = \mathcal{V}_I^+ \cup \mathcal{V}_S^+$ the variables in this extended language. Define the smallest canonical model for HCL

$$\mathcal{M}_* = \langle W_*, N_*, D_*, I_*, \sigma_*\rangle$$

as follows. Let $MAX(\Gamma)$ indicate that the set Γ is an HCL-maximally consistent set of formulas of L_1^+,

- $W_* = \{\Gamma \mid MAX(\Gamma)$ and Γ has the $\forall$-property$\}$
- $X \in N_*(\Gamma, U)$ iff there is an $[\{x\}\psi]\varphi \in \Gamma$ such that,
  - $X = \{\Delta \in W_* \mid \varphi \in \Delta\}$
  - $U = \{d \in D_* \mid \mathcal{M}_*, \Gamma \models_\sigma \psi[d/x]\}$
- $D_* = \mathcal{V}_I^+$
- $\langle x_1, \ldots, x_n\rangle \in I_*(\varphi, \Gamma)$ iff $\varphi(x_1, \ldots, x_n) \in \Gamma$
- For every individual variable $x \in \mathcal{V}_I^+, \sigma_*(x) = x$
- For every set variable $X \in \mathcal{V}_S^+, \sigma_*(X, \Gamma) = \{u \mid Xu \in \Gamma\}$

The definition of the neighborhood function N_* essentially says that a set of states of the canonical model is necessary at a state Γ

[10] A frame is supplemented if the corresponding neighborhood function satisfies *outcome monotonicity* as reported in Def 2.

[11] $COMP$. stands for "comprehension" by analogy with the comprehension axiom of set theory.

precisely when Γ claims that it should, notice that this definition is not circular but recursive and well-founded [12].

For any formula $\varphi \in \mathcal{L}_1$, let $|\varphi|_*$ be the proof set of φ w.r.t. model $\mathcal{M}_*$, that is,

$$|\varphi|_* = \{\Gamma \mid \Gamma \in W_* \textit{ and } \varphi \in \Gamma\}$$

The fact that N_* is a well-defined function follows from the fact that Γ contains the rules RE and $NORM$.

Definition 7 *Let $\mathcal{M} = \langle W, N, D, I, \sigma\rangle$ be any first-order constant domain coalition model. $\mathcal{M}$ is said to be **canonical for** HCL provided $W = W_*$, $D = D_*$, $I = I_*$, $\sigma = \sigma_*$ and*

$$|\varphi|_* \in N(\Gamma, U) \textit{ iff } [\{x\}\psi]\varphi \in \Gamma \textit{ with}$$

$$U = \{d \in D_* \mid \mathcal{M}_*, \Gamma \models_\sigma \psi[d/x]\}$$

Thus the model $\mathcal{M}_*$ is the smallest canonical model for HCL.

Lemma 2 *(Truth Lemma) For each $\Gamma \in W_*$ and formula $\varphi \in \mathcal{L}_1$,*

$$\varphi \in \Gamma \textit{ iff } \mathcal{M}_*, \Gamma \models_{\sigma_*} \varphi$$

Proof. The proof is by induction on φ. The base case and propositional connectives are as usual. The 'if' direction of the modal and quantifier case is straightforward. We will discuss the two cases. Suppose that $\forall x\varphi(x) \notin \Gamma$ (where x is an individual variable). Then since Γ is maximal, $\neg\forall x\varphi(x) \in \Gamma$, and so by the $\forall$-property there is some variable $y \in \mathcal{V}_I^+$ such that $\neg\varphi[y/x] \in \Gamma$, and so $\varphi[y/x] \notin \Gamma$. Thus by induction hypothesis, $\mathcal{M}_*, \Gamma \not\models_{\sigma_*[y/x]} \varphi(x)$. Hence $\mathcal{M}_*, \Gamma \not\models_{\sigma_*} \forall x\varphi(x)$.

For a set variable X we proceed similarly, suppose that $\forall X\varphi(X) \notin \Gamma$. Then since Γ is maximal, $\neg\forall X\varphi(X) \in \Gamma$, and so by the $\forall$-property there is some variable $Y \in \mathcal{V}_S^+$ such that $\neg\varphi[Y/X] \in \Gamma$, and so $\varphi[Y/X] \notin \Gamma$. Thus by induction hypothesis, $\mathcal{M}_*, \Gamma \not\models_{\sigma_*[Y/X]} \varphi(X)$. Hence $\mathcal{M}_*, \Gamma \not\models_{\sigma_*} \forall X\varphi(X)$.

Regarding the modal case, the proof proceeds by construction of N_* and definition of truth: $[\{x\}\psi]\varphi \in \Gamma$ iff (by construction) $|\varphi| \in N_*(\Gamma, U)$[13] iff (by definition of truth) $\mathcal{M}_*, \Gamma \models_{\sigma_*} [\{x\}\psi]\varphi$ □

We now have to check that the canonical model $\mathcal{M}_*$ is weakly playable.

Lemma 3 *The canonical model $M_* = \langle W_*, N_*, D_*, I_*, \sigma_*\rangle$ is weakly playable*

Proof. We have to check that the properties in Definition 4 are canonical. For instance consider Condition 6 (superadditivity). By contradiction, suppose $C_1 \cap C_2 = \emptyset$, $X_1 \in N_*(w, C_1)$, $X_2 \in N_*(w, C_2)$ and $X_1 \cap X_2 \notin N_*(w, C_1 \cup C_2)$. In what follows $C_i(x)$ $(i = 1, 2)$ corresponds to an HCL formula such that

$$C_i = \{d \in D_* \mid \mathcal{M}_*, w \models_\sigma C_i(d)\}$$

Notice that for every subset C of the domain D^* such that $N_*(w, C)$ it is always possible to find a formula $\varphi(x)$ such that its extension corresponds to C[14]. We then have that $[\{x\}C_1(x)]\varphi_1$, $[\{x\}C_2(x)]\varphi_2 \in w$ for $|\varphi_1|_* = X_1$ and $|\varphi_2|_* = X_2$.Then, because $C_1 \cap C_2 = \emptyset$, by axiom S. we have that $[\{x\}C_1(x) \vee C_2(x)](\varphi_1 \wedge \varphi_2) \in w$ and since $|\varphi_1 \wedge \varphi_2|_* = X_1 \cap X_2$ we have $X_1 \cap X_2 \in N(w, C_1 \cup C_2)$, which is a contradiction.

□

[12] This can be intuitively seen via an iterative construction on the modal depth of ψ in $[\{x\}\psi]\varphi$.

[13] With $U = \{d \in D_* \mid \mathcal{M}_*, \Gamma \models_{\sigma_*} \psi[d/x]\}$.

[14] This holds by construction of N_* in the definition of the smallest canonical model.

Theorem 2 *For any canonical model $\mathcal{M}$ for HCL, φ is valid in the canonical model iff $\vdash \varphi$.*

Now, from the truth lemma, via a standard argument we get as corollary:

Corollary 1 *The class of all (weakly playable) constant domain coalition frames is sound and complete for HCL*

7 Translating Quantified Coalition Logic

In this section we show how CL and QCL can be embedded into HCL. We prove this by introducing the following translation τ defined as follows,

Definition 8 *Let τ be a function from propositional logic to HCL defined as follows:*

$$\begin{array}{rcl} \tau(\top) & = & \top \\ \tau(p) & = & T(p) \\ \tau(\neg\varphi) & = & \neg(\tau(\varphi)) \\ \tau(\varphi_1 \vee \varphi_2) & = & (\tau(\varphi_1) \vee \tau(\varphi_2)) \end{array}$$

where $\top$ is any tautology, $T(p)$ is a propositional atom which mimic the boolean variable p.

The translation τ can now be extended to handle CL modal operators as follows

$$\tau([C]\psi) = [\{x\}C(x)]\tau(\psi)$$

where, for every coalition $C \in Ag$ let $C(x)$ be a monadic first-order predicate such that $d \in C$ iff $C(d)$ holds[15]. Regarding QCL, we first define a translation for coalition predicates,

Definition 9 *Let δ be a function from QCL coalition predicates and HCL, defined as follows:*

$$\begin{array}{rcl} \delta(X, subseteq(C)) & = & \forall x(Xx \rightarrow C(x)) \\ \delta(X, supseteq(C)) & = & \forall x(C(x) \rightarrow Xx) \\ \delta(X, \neg P) & = & \neg\delta(X, P) \\ \delta(X, P_1 \vee P_2) & = & \delta(X, P_1) \vee \delta(X, P_2) \end{array}$$

We then extend τ as follows:

$$\begin{array}{rcl} \tau(\langle P\rangle\varphi) & = & \exists X((\delta(X, P)) \wedge [\{x\}(Xx)]\tau(\varphi)) \\ \tau([P]\varphi) & = & \forall X((\delta(X, P)) \rightarrow [\{x\}(Xx)]\tau(\varphi)) \end{array}$$

Theorem 3 *Let $\mathcal{M}^* = \langle S, E, \pi\rangle$ be a CL model, s a state and φ a CL formula then there exists a HCL model $\mathcal{M} = \langle W, N, D, I, \sigma\rangle$ such that*

- $\mathcal{M}^*, s \models \varphi$ *iff* $\mathcal{M}, s \models_\sigma \tau(\varphi)$

Proof. By induction on φ, let $\mathcal{M} = \langle W, N, D, I, \sigma\rangle$ be defined as follows:

- $W = S$; $N = E$; $D = Ag$
- $I(T, s) = \{p \mid p \in \pi(s)\}$
- $I(C, s) = \{d \mid d \in C\}$, for all $C \subseteq Ag$
- $\sigma(p) = p$, for all boolean variables $p \in \phi_0$

1. Suppose $\mathcal{M}^*, s \models p$, where p is atomic, by the definition of $\mathcal{M}^*$ it easy to see that $\mathcal{M}, s \models T(p)$.

[15] Notice that it is always possible to define such a predicate because the set of Ag is finite and bounded.

2. Suppose $\mathcal{M}^*, s \models [C]\psi$, we have to prove that
$$\mathcal{M}, s \models_\sigma [\{x\}C(x)]\psi$$
By induction hypothesis, we know that there exists an HCL model $\mathcal{M}$ such that
$$\mathcal{M}^*, s \models \psi \text{ iff } \mathcal{M}, s \models_\sigma \tau(\psi).$$
From the definition of $\mathcal{M}$ we have that
$$C = \{d \mid M, s \models_{\sigma[d/x]} C(x)\}$$
then because $N = E$ we have
$$\mathcal{M}, s \models_\sigma [\{x\}C(x)]\psi$$

other cases are similar. □

Theorem 4 *Let $\mathcal{M}^* = \langle S, E, \pi, \rangle$ be a QCL model, s a state and φ a QCL formula, then there exists a HCL model $\mathcal{M} = \langle W, N, D, I \rangle$ and a σ such that*

- $\mathcal{M}^*, s \models \varphi$ *iff* $\mathcal{M}, s \models_\sigma \tau(\varphi)$

Proof. Similar to Theorem 3. □

8 Conclusions

We introduce higher-order coalition logic, a monadic second-order (multi-)modal logic. In particular, we define a coalition logic in which agents can affect themselves by collapsing the language to talk about coalitions and the language to talk about states. We introduce a formalism which permits a very general way to quantify over coalitions. We provide an expressive and compact way to represent coalitions by means of the set-binding operator $\{x\}\varphi$.

We give an axiomatization and prove completeness. We strengthen the usefulness of employing neighborhood semantics by analyzing relationships between BF and CBF axiom schemas. In general, HCL is undecidable[16]. However there is room for further research in identifying a proper decidable fragment of HCL, in fact we know that CL has the finite model property [18], the group operator $\{x\}\psi$ does not extends the complexity of MSO [12] and the interaction axioms in HCL are limited to CBF.

The results presented provide a general, expressive and formal framework to represent knowledge about extensive games with perfect information with simultaneous actions and to further study higher-order coalition logics.

In [1] the possibility to quantify over coalitions by using a first-order apparatus is considered unfeasible, in fact it is argued that explicit quantification over sets leads to undecidability over infinite domains, and very high computational complexity even over finite domains, which would make the logic too computationally complex to be of practical interest. We disagree with this conclusion, in fact, it is true that MSO is a proper fragment of full second-order logic that already has the full complexity. Nevertheless as reported in [20], MSO is in some respect much better behaved than full second-order logic. In particular Rabin's theorem tell us that the satisfiability problem for MSO with respect to countably branching trees is decidable, unlike the corresponding problem for full second-order logics. Moreover MSO is deeply studied as successful practical query formalism on finite structures and XML documents [21].

[16] Think for instance about general infinite domains.

A promising direction would be to formalize different fragments of HCL by constraining both the domain and the neighborhood structure in order to get nice computational properties. Further research must be done in order to assess computational properties of HCL.

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