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LP Solvable Models for Multiagent Fair Allocation Problems

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Abstract. This paper proposes several operational approaches for solving fair allocation problems in the context of multiagent optimization. These problems arise in various contexts such as assigning conference papers to referees or sharing of indivisible goods among agents. We present and discuss various social welfare functions that might be used to maximize the satisfaction of agents while maintaining a notion of fairness in the distribution. All these welfare functions are in fact non-linear, which precludes the use of classical min-cost max-flow algorithms for finding an optimal allocation. For each welfare function considered, we present a Mixed Integer Linear Programming formulation of the allocation problem that can be efficiently solved using standard solvers. The results of numerical tests we conducted on realistic cases are given at the end of the paper to confirm the practical feasibility of the proposed approaches.

1 Introduction

Allocation problems are pervasive in the field of multiagent decision making. The general problem consists of allocating m items to n agents. Depending on the context, the items can represent tasks, resources, goods or any object that can be assigned to one or several agents. In practical applications, one can distinguish different variants of the general problem: 1) one to one allocation problems, also known as *assignment problems* [5] where $m = n$ and for which a single item is assigned to any agent and vice versa; 2) many to one allocation problems where m is greater than n and for which several items can be assigned to each agent (e.g. sharing indivisible goods [3], the Santa Claus Problem [1]); 3) many to many allocation problems where items (tasks) can be assigned in parallel to several agents, each agent being in charge of possibly several tasks. A typical example of the latter case is the conference paper allocation problem where items are papers to be reviewed (k times each) and agents are referees [12, 9]. In classical formulations of these problems, preferences are supposed to be additively decomposable in two ways. On the one hand, the Social Welfare function measuring the overall utility of a solution (allocation) for the collection of agents is defined as the sum of individuals' utilities (utilitarian approach). On the other hand, individual utility functions are supposed to be additive, i.e. the value of a subset of items for a given agent is defined as the sum of the utilities of each item. Hence the goal is to maximize an overall linear function of type $\sum_{i,j} u_{ij} z_{ij}$ where u_{ij} (resp. and z_{ij}) represents, for any item j and agent i, the utility of object j for agent i (resp. a boolean decision variable concerning the assignment of j to i). This linear function is typical of the so-called "linear assignment problem" that can be solved in polytime either by the hungarian algorithm (for one to one assignment problems), or by algorithms designed for min-cost max-flow problems, and more generally by integer linear programming (for more details see [13, 5]).

However, the double additivity of the overall utility function can be questioned. Firstly, linear social welfare functions do not provide any control on the fairness of the solution. The compensative nature of the aggregation obtained by a sum of individual utilities allows serious inequalities in the repartition of satisfactions. This might be a drawback in many decision making processes involving multiple agents. This is the case in allocation procedures where fairness is often an important issue. For example, in the paper assignment problems, seeking the satisfaction of all reviewers will promote their good cooperation for similar tasks in the next conference. In resource allocation problems, avoiding important ruptures in services requested by every client will preserve their satisfaction and contribute to the development of the service. The following example illustrates the idea of fairness in multiagent allocation problems:

Example 1 *We consider a conference paper allocation problem with 5 papers that must be assigned to 3 reviewers in such a way that each paper gets exactly two reviews and each reviewer gets at most 4 papers. In a preliminary round, reviewers have expressed their willingness to review with respect to the different papers using utility scale* $\{1, \ldots, 4\}$, 4 *being the most favorable evaluation (we assume here that reviewers enjoy referring papers; whenever reviewing a paper is seen as a charge, utilities can be replaced by costs). The utilities are given in the following table:*

u_{ij}	*Paper 1*	*Paper 2*	*Paper 3*	*Paper 4*	*Paper 5*
Reviewer 1	*3*	*3*	*4*	*3*	*4*
Reviewer 2	*3*	*4*	*4*	*2*	*3*
Reviewer 3	*1*	*2*	*3*	*2*	*3*

In this problem, any solution is characterized by a 3×5 *boolean matrix Z whose general term* $z_{ij} = 1$ *if and only if reviewer* i *receives paper* j*. This implicitly represents a set of* 2^{15} *solutions but many of them do not satisfy the constraints characterizing a feasible allocations. If we want to find a feasible allocation maximizing the sum of individual utilities we have to solve the following linear program:*

$$(\mathcal{P}_0) \quad \max \sum_{i=1}^{3} \sum_{j=1}^{5} u_{ij} z_{ij} \quad s.t. \left\{ \begin{array}{lcll} \sum_{i=1}^{3} z_{ij} & = & 2 & \forall j = 1 \ldots 5 \\ \sum_{j=1}^{5} z_{ij} & \leq & 4 & \forall i = 1 \ldots 3 \\ z_{ij} \in \{0,1\} & & & \forall i, \forall j \end{array} \right.$$

The optimal solution is as follows: reviewer 1 receives papers $\{1, 3, 4, 5\}$*, reviewer 2 receives* $\{1, 2, 3\}$ *and reviewer 3 receives* $\{2, 4, 5\}$*. The overall utility of this solution is 32 which can be decomposed into 3 components to make explicit the reviewers' satisfaction profile. This gives* $(14, 11, 7)$ *which is quite unfair, one agent*

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getting two times more than another one. Such inequalities are neither desirable nor necessary. For example, if we consider another repartition such as that one: reviewer 1 gets $\{1,4,5\}$*, reviewer 2 gets* $\{1,2,3\}$ *and reviewer 3 gets* $\{2,3,4,5\}$ *we obtain a better balanced utility profile:* $(10,11,10)$ *for a very small reduction of the average satisfaction (*$31/3$ *instead of* $32/3$*). Although more attractive in terms of fairness, this solution cannot be found by solving* $(\mathcal{P}_0)$ *because it is suboptimal.*

This example shows that the linear assignment formulation is perhaps not suited to multiagent optimization problems for which achieving a well-balanced utility profile is important. Moreover, in some cases, there exist positive or negative synergies among items that cannot be represented by an additive function. When some items are complementary, the value of the set is more than the sum of its parts. On the contrary, when items are redundant, resorting to subadditive utility functions might be necessary. This is another reason why additive utilities are not always relevant in allocation problems. In this paper, we concentrate on the first problem: the determination of fair allocations in multiagent problems. In order not to multiply the sources of complexity in the same study, we assume here that individual utility functions are additive (as in the classical case) but we will resort to non-linear social welfare functions able to capture an idea of fairness in the evaluation of solutions. These models are imported from Social Choice Theory (inequality measurement) and multicriteria analysis (compromise search). The measurement of inequalities has indeed received much attention in Social Choice Theory and Mathematical Economics where several non-linear social welfare functions have been proposed to capture an idea of fairness in the evaluation of solutions (see e.g. [16, 17, 24]). Other models have been developed in multicriteria analysis for characterizing good compromise solutions with respect to conflicting objective (e.g. [27]). The aim of this paper is to investigate the use of such models in multiagent combinatorial optimization problems. We consider allocation problems with various non-linear utility functions and propose reformulation that can be solved by standard linear programming solvers for real-size instances. This study concerns the case of centralized information. For a distributed version of the multiagent allocation problem, the reader should consult [8, 7, 15].

The paper is organized as follows. In Section 2 we discuss approaches focusing on the least satisfied agent. In Section 3 we consider Gini social evaluation functions that make possible to control the weight of any agent, depending on its rank in the satisfaction order. In Section 4 we consider specific instances of Choquet integrals that allow to favor well balanced utility profiles while keeping the possibility of attaching a specific weight to each agent in each coalition. For all these models we provide solution methods using linear-programming. The practical tests performed to illustrate the effectiveness of these models are given in Section 5.

2 Basic formulations

The general allocation problem we are considering can be stated as follows: we want to distribute m items over n agents. The number of items that can be allocated to agent i is restricted to interval $[\alpha_i, \beta_i]$, $i = 1,\dots,n$. Item j must be assigned to a number of agents restricted to the interval $[\alpha'_j, \beta'_j]$, $j = 1,\dots,m$. A $n \times m$ matrix gives the utility u_{ij} of assigning item j to agent i. Hence, denoting z_{ij} the allocation variable for agent i and item j we obtain the following 0-1 optimization problem:

$$\max \ \psi(x_1,\dots,x_n) \tag{1}$$

$$(\Pi_\psi) \qquad s.t. \begin{cases} x_i = \sum_{j=1}^m u_{ij} z_{ij} & i = 1,\dots,n \\ \alpha_i \le \sum_{j=1}^m z_{ij} \le \beta_i & i = 1,\dots,n \\ \alpha'_j \le \sum_{i=1}^n z_{ij} \le \beta'_j & j = 1,\dots,m \end{cases} \tag{2}$$

$$z_{ij} \in \{0,1\} \ \forall i, \forall j \tag{3}$$

where ψ is a social welfare function defined from individual satisfaction indices $x_i, i = 1,\dots,n$ as a non-decreasing function of its arguments. This general optimization program fits to many different situations involving multiple agents. For example, in fair allocation of indivisible goods, we set $\alpha_i = 0$ and $\beta_i = m$ for $i = 1\dots n$ and $\alpha'_j = \beta'_j = 1$, $j = 1\dots m$. This formulation also fits to conference paper allocation problems. In this case $\alpha_i = 0$ and $\beta_i = K$, $i = 1\dots n$, where K is the maximal number of papers that can be allocated to a reviewer, and $\alpha'_j = \beta'_j = R$, $j = 1\dots, m$ (a paper must be reviewed by R referees).

In the introduction, we have seen that linear combinations of individual utilities do not properly capture the idea of fairness of a solution; this suggests resorting to non-linear functions for ψ. In this direction, the maxmin approach that consists of maximizing $\psi(x_1,\dots,x_n) = \min_{i=1\dots n}\{x_i\}$ is probably the simplest alternative to the linear model. This criterion that directly translates the idea of Economic Egalitarianism consists of maximizing the satisfaction of the least satisfied agent. Coming back to Example 1, we can see that the initial allocation yielding $(14,11,7)$ as utility vector would be suboptimal since the other solution presented yields $(10,11,10)$ which guarantees a better worst-case value (10 instead of 7). Unfortunately problem $\Pi_{\min}$ (maxmin allocation problem) is NP-hard as soon as there are two agents [4, 9, 11]. Although min is not a linear function, problem $\Pi_{\min}$ can be formulated as a 0-1 linear program as follows:

$$\begin{aligned} & \max \ z \\ (\Pi'_{\min}) \qquad & s.t. \begin{cases} z \le \sum_{j=1}^m u_{ij} z_{ij} & i = 1,\dots,n \\ \alpha_i \le \sum_{j=1}^m z_{ij} \le \beta_i & i = 1,\dots,n \\ \alpha'_j \le \sum_{i=1}^n z_{ij} \le \beta'_j & j = 1,\dots,m \end{cases} \\ & z_{ij} \in \{0,1\} \ \forall i, \forall j \end{aligned}$$

This makes it possible to solve realistic-size instances using standard solvers as will be shown in Section 5.

Focusing on the worst case is sometimes too drastic because the smaller component of utility vectors might mask very different situations. There are indeed undesirable drowning effects with the min that prevent discrimination between two utility vectors such as (10, 10, 10) and $(10,20,20)$ for example. To overcome the problem we consider instead a refinement of the min with a weighted sum:

$$\psi(x_1,\dots,x_n) = \min_{i=1\dots n} x_i + \varepsilon \sum_{i=1}^n x_i \tag{4}$$

where ε is a strictly positive real number, chosen arbitrary small. This criterion can be seen as a particular instance of weighted Tchebycheff distance with respect to a reference point, a classical scalarizing function used to generate compromise solutions in multiobjective optimization [27]. It can also be seen as a lexicographic aggregation of the egalitarian criterion (min) with the utilitarian criterion (sum of utilities) with priority to egalitarianism. Obviously, this augmentation of min does not change the complexity of the problem nor the existence of a linear reformulation.

However, using an augmented min does not really solve the problem but shifts it to other components than the minimum. For example we cannot discriminate between (10, 10, 10, 40) and (10, 20, 20, 20).

To overcome the problem, another possibility is the leximin criterion. If $x^\uparrow$ represents the vector x whose components have been sorted by increasing order ($x^\uparrow_i \leq x^\uparrow_{i+1}$) then x is preferred to y according to the leximin if and only if $x^\uparrow_k > y^\uparrow_k$ for some k and $x^\uparrow_i = y^\uparrow_i$ for all $i < k$. Thus (10, 20, 20, 20) is preferred to (10, 10, 10, 40). The problem of finding a leximin-optimal allocation is proved NP-hard in [9]. Nevertheless, the linearization of the min can be adapted for the leximin operator using lexicographic linear optimization as shown in [30] and [18]. We introduce now more compensatory inequality measures used in Social Choice Theory.

3 Generalized Gini social-evaluation functions

Generalized Gini social-evaluation functions as defined in Social Choice theory by Blackorby and Donaldson [2] can be an interesting choice for ψ in (1) for inequality measurement. They are defined as follows:

$$W(x) = \sum_{i=1}^{n} w_i x^\uparrow_i \tag{5}$$

where $w_i > w_{i+1}$ for $i = 1, \dots, n-1$. Note that if $w_1 = 1$ and the other weights tend to 0 then $W(x)$ tends to the egalitarian criterion (min). This shows that finding a W-optimal allocation is also NP-hard. Whenever differences of type $w_i - w_{i+1}$ tends to 0 (i.e. weights tend to be nearly equal) $W(x)$ tends to the utilitarian criterion (sum). On the contrary when differences of type $w_i - w_{i+1}$ tend to be arbitrarily large, then $W(x)$ tends to the leximin criterion. Of course $W(x)$ offers many other possibilities. This family of functions is also known in multicriteria analysis under the name of *ordered weighted averages* (OWA) [31]. In the field inequality measurement, generalized Gini social-evaluation functions have received an axiomatic justification by Weymark [29]. His axiomatic analysis shows that W has several nice properties including monotonicity with respect to each component (here individual utilities) which ensures Pareto-efficiency of W-optimal solutions, but also monotonicity with respect to utility transfers from a "richer" agent to a "poorer" agent, which guarantees the fairness of W-optimal solutions. This means that transfers reducing inequalities, also known as *Pigou-Dalton transfers* [17], will improve the value of social utility $W(x)$. More formally, let $x \in \mathbb{R}^n_+$ be the utility vector of a feasible solution, such that $x_i > x_j$ for some i, j in a multiagent problem, then for any other feasible solution yielding an utility vector of the form $y = (x_1, \dots, x_j + \varepsilon, \dots, x_i - \varepsilon, \dots, x_n)$ with $0 < \varepsilon < x_i - x_j$, we have $W(y) \geq W(x)$. In mathematical terms, this means that $W(x)$ is Schur-concave [16]. Interestingly enough, function $W(x)$ can be rewritten as follows:

$$W(x) = \omega . L(x) \tag{6}$$

where $\omega = (w_1 - w_2, w_2 - w_3, \dots, w_{n-1} - w_n, w_n)$ is a positive weighting vector and $L(x) = (L_1(x), \dots, L_n(x))$ is the Lorenz vector associated to x defined by $L_k(x) = \sum_{i=1}^{k} x^\uparrow_i$. The notion of Lorenz vector was initially used to measure relative inequalities of vectors having the same average (see the results of Hardy, Little-Hood and Polya in [16]). Lorenz vectors can also be used to measure inequalities of vectors having possibly different means, using the *generalized Lorenz dominance* introduced by Shorrocks [26].

Definition 1 *Generalized Lorenz dominance is a strict preference relation $\succ_L$ defined on utility vectors in $\mathbb{R}^n_+$ by: $x \succ_L y$ if $L_i(x) \geq L_i(y)$ for all $i = 1 \dots n$, one of these inequalities being strict.*

For example, in Figure 1 the line separating light gray and dark gray areas represents the Lorenz curve $L_i(x), i = 1, \dots, n$ of a utility vector x unequally dividing 100 utility points among 10 agents. On the same figure the diagonal line bounding above the light gray area represents the Lorenz curve of an the ideal distribution $y = (10, \dots, 10)$ such that $L^*_i(y) = 10\, i$ for $i = 1, \dots, 10$. We can see that the former line remains below the diagonal line which shows that x is Lorenz-dominated by y.

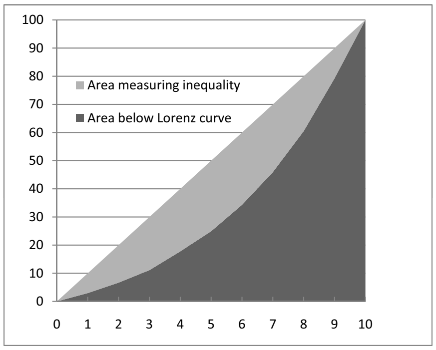

Figure 1. Lorenz curve and Gini index

Lorenz-dominance is a partial weak-order comparing utility vectors in terms of fairness. However, due to the incompleteness of this model, it cannot be used easily in fair multiagent optimization problems. Fortunately, Equation (6) shows that the generalized Gini social-evaluation function $W(x)$ induces a linear extension of the Generalized Lorenz Dominance partial order, better suited to optimization tasks. Let us consider the following example:

Example 2 *Consider a problem involving 3 agents and assume we have to compare 3 feasible solutions with utility vectors $x = (11, 12, 13)$, $y = (9, 12, 14)$ and $z = (17, 15, 8)$. We have $L(x) = (11, 23, 36)$, $L(y) = (9, 21, 35)$ and $L(z) = (8, 23, 40)$. $L(x)$ Pareto-dominates $L(y)$ which means that $x \succ_L y$. Moreover no dominance holds between $L(z)$ and the two other Lorenz vectors which leaves z uncomparable. If we use a Gini social-evaluation function $W(x)$ with $w_1 = 5/9, w_2 = 3/9$ and $w_1 = 1/9$ we get: $W(x) = 104/9, W(y) = 95/9$ and $W(z) = 102/9$ which entails the following preference order: $x \succ z \succ y$.*

As pointed out by Weymark [29], $W(x)$ is a generalization of the original *Gini social-evaluation function* defined by:

$$\mathcal{W}(x) = \frac{1}{n^2} \sum_{i=1}^{n} (2(n-i)+1) x^\uparrow_i \tag{7}$$

This social function is at the origin of the so-called *Gini coefficient* [10] measuring the degree of inequality of an income distribution in a society and defined by $I(x) = 1 - \mathcal{W}(x)/\mu(x)$ where $\mu(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$. This index actually measures a "distance" to perfect equity that can be illustrated on Figure 1. It can indeed be shown that $I(x)$ equals two times the area in light gray. It is important to note that, despite its relevance for measuring inequalities, Gini coefficient $I(x)$ cannot be used directly for ψ in (1) because it does not satisfy strict monotonicity with respect to Pareto dominance. Indeed, scaling all incomes (here individual utilities) proportionally does not affect the value of the index. Hence nothing guarantees the Pareto efficiency of I-optimal solutions. This is the reason why we shall use $\mathcal{W}(x)$ and not $I(x)$ in (1), or more generally any instance of $W(x)$, to determine fair Pareto-efficient allocations.

We present now a LP-solvable formulation of problem Π_ψ with $\psi(x) = W(x)$ in Equation (1). Following an idea introduced in [20], we express the k^{th} Lorenz component $L_k(x)$, for any fixed x, as the solution of the following linear program:

$$(\mathcal{P}_{L_k}) \qquad \begin{array}{l} \min \ \displaystyle\sum_{i=1}^{n} a_{ik}x_i \\ s.t. \left\{ \begin{array}{ll} \sum_{i=1}^{n} a_{ik} = k & \\ a_{ik} \leq 1 & i = 1 \dots n \end{array} \right. \\ a_{ik} \geq 0 \ \ i = 1 \dots n \end{array}$$

$L_k(x)$ can also be obtained by solving the dual problem:

$$(\mathcal{D}_{L_k}) \qquad \begin{array}{l} \max \ kr_k - \displaystyle\sum_{i=1}^{n} b_{ik} \\ s.t. \left\{ \ r_k - b_{ik} \leq x_i \quad i = 1 \dots n \right. \\ b_{ik} \geq 0 \ \ i = 1 \dots n \end{array}$$

Using Equation (6) and the fact that components of ω are strictly positive, we can combine problems $(\mathcal{D}_{L_k})$ $k = 1 \dots n$ with the initial allocation problem Π_ψ to get the following linear formulation [20, 11]:

$$(\Pi'_W) \qquad \begin{array}{l} \max \ \displaystyle\sum_{k=1}^{n} \omega_k (kr_k - \sum_{i=1}^{n} b_{ik}) \\ s.t. \left\{ \begin{array}{ll} \alpha_i \leq \sum_{j=1}^{m} z_{ij} \leq \beta_i & i = 1, \dots, n \\ \alpha'_j \leq \sum_{i=1}^{n} z_{ij} \leq \beta'_j & j = 1, \dots, m \\ r_k - b_{ik} \leq \sum_{j=1}^{m} u_{ij} z_{ij} & i, k = 1 \dots n \end{array} \right. \\ b_{ik} \geq 0 \ \ i, k = 1 \dots n \\ z_{ij} \in \{0, 1\} \ \ i = 1 \dots n, j = 1 \dots m \end{array}$$

Remark that using a similar combination with primal problems $(\mathcal{P}_{L_k})$ instead of $(\mathcal{D}_{L_k})$, $k = 1 \dots n$ leads to a quadratic function due to products of variables such as $a_{ik}x_i$. Fortunately, resorting to the dual problems preserves linearity of the objective function. Hence we get a linear problem with $n^2 + 2(m+n)$ constraints, nm boolean variables, and $n^2 + n$ continuous variables, which can be solved with standard LP-solvers as will be shown in Section 5.

4 The Choquet integral as an inequality measure

Until now, all social evaluation functions we have considered for ψ in Equation (1) are symmetric. This means that the value $\psi(x)$ remains unchanged by permutation of the components of x. This property basically says that every agent has the same importance in the evaluation process, no matter who he really is, only his satisfaction level is considered. This anonymity property is generally seen as desirable in multiagent decision making. However, in particular situations, it might happen that some agents are more important than others. This is the case for example in resource allocations problems where clients have exogenous rights (see for example [3, 19]). This suggests considering weighted extensions of social evaluation functions. Actually Gini social evaluation functions can easily be extended to incorporate weights of agents. For example, one can complete the initial population of agents with clones of initial agents whose multiplicity is proportional to the agents' weights. It is then sufficient to apply function $W(x)$ on augmented utility vectors. This simple idea can be implemented without resorting to an explicit multiplication of agents. It is more appropriate to consider weighted extensions of ordered weighted averages named WOWA for weighted ordered weighted averages [28]. Such WOWA operators can be linearized similarly as $W(x)$ as shown by Ogryczak in [21]. This can be used to produce a LP-solvable formulation of Π_ψ where ψ is an WOWA, as done with Π'_W proposed in Section 3 for $\psi = W$.

One step further in the sophistication of social evaluation functions, we might be interested in extending inequality measures to tackle situations where the importance attached to a group of agents cannot simply be represented by the sum of their weights. The introduction of a non-additive measure to model the importance of coalitions is classical in Game Theory where it is used to model positive or negative synergies among players. We recall now some definitions linked to capacities, a classical tool to model the importance of coalitions within the set $N = \{1, \dots, n\}$ of agents.

Definition 2 *A capacity is a mapping $v : \mathcal{P}(N) \to \mathbb{R}$ such that $v(\emptyset) = 0$, $v(N) = 1$, and $v(A) \leq v(B)$ whenever $A \subseteq B$.*

Definition 3 *A capacity v is said to be* convex *when $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$ for all $A, B \subseteq N$, and* additive *when $v(A \cup B) + v(A \cap B) = v(A) + v(B)$ for all $A, B \subseteq N$.*

The *dual* $\bar{v}$ of a capacity v is a capacity defined by $\bar{v}(A) = 1 - v(N \backslash A)$ for all $A \subseteq N$. It is easy to see that $\bar{\bar{v}} = v$. Moreover, when v is convex we have $v(A) + v(N \backslash A) \leq 1$, hence by definition of $\bar{v}$ we have $v(A) \leq \bar{v}(A)$. Hence the notion of core can be introduced:

Definition 4 *The core of a capacity v is defined by:*

$$core(v) = \{\lambda \in \Lambda : v(A) \leq \lambda(A) \leq \bar{v}(A)\} \tag{8}$$

where Λ is the set of additive capacities defined on $\mathcal{P}(N)$.

A well-known result due to Shapley [25] is that any convex capacity has a non-empty core. This property will play a crucial role later in the paper. Any capacity is completely characterized by 2^n coefficients, representing the importance $v(A)$ of any coalition $A \subseteq N$. When v is additive it admits a very compact representation using only n coefficients $v(\{i\}), i = 1 \dots n$ since $v(A) = \sum_{i \in A} v(\{i\})$, but this is to the detriment of expressivity since no synergy is allowed among agents. In the general case, a capacity admits an alternative representation named the Möbius inverse:

Definition 5 *To any capacity $v : \mathcal{P}(N) \to \mathbb{R}$ a mapping $m : \mathcal{P}(N) \to \mathbb{R}$ called* Möbius inverse *can be associated, defined by:*

$$\forall A \subseteq N, m(A) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} v(B) \tag{9}$$

v can be reconstructed from its Möbius inverse as follows:

$$\forall A \subseteq N, v(A) = \sum_{B \subseteq A} m(B) \tag{10}$$

Using the Möbius inverse, we can define the notion of *k-additive* capacities as follows [14]:

Definition 6 *A capacity is said to be k-additive when its Möbius inverse vanishes for any $A \subseteq N$ such that $|A| > k$, and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$. More formally:*

$$(i) \ \ \forall A \subseteq N, |A| > k \Rightarrow m(A) = 0$$
$$(ii) \ \ \exists A \subseteq N, |A| = k \text{ and } m(A) \neq 0$$

If $k = 1$ we get an additive capacity. $k-$additive capacities for small values of k greater than 1 are very useful because in practical situations, they offer a sufficient expressivity to model positive or negative interactions between agents with a reduced number of parameters. For example, when $k = 2$ the capacity is completely characterized by $(n^2+n)/2$ coefficients (one Möbius mass for every singleton and every pair).

In decision theory the main model based on the use of a capacity is called the Choquet integral [23]. The Choquet integral of a utility vector $x \in \mathbb{R}^n$ with respect to capacity v is defined by:

$$C_v(x) = \sum_{i=1}^{n}[x_i^{\uparrow} - x_{i-1}^{\uparrow}]v(X_i^{\uparrow}) = \sum_{i=1}^{n}[v(X_i^{\uparrow}) - v(X_{i+1}^{\uparrow})]x_i^{\uparrow} \quad (11)$$

where $x_0^{\uparrow} = 0$ and $X_i^{\uparrow}$ is the set of the $n - i + 1$ most satisfied agents, formally $X_i^{\uparrow} = \{j \in N, x_j \geq x_i^{\uparrow}\}$ for $i \leq n$. The left part of (11) has the following meaning: $X_1^{\uparrow}$ contains all agents and they are at least satisfied to level $x_1^{\uparrow}$; so we start by multiplying $x_1^{\uparrow}$ by $v(X_1^{\uparrow})$; then every agent in $X_2^{\uparrow}$ gets at least an utility increment of $(x_2^{\uparrow} - x_1^{\uparrow})$ so that we add the coefficient $(x_2^{\uparrow} - x_1^{\uparrow})$ weighted by the importance of coalition $X_2^{\uparrow}$, and so on .... When used with a capacity such that $v(N) = 1$ and $v(A) = 0$ for all $A \neq N$ then $C_v(x) = x_1^{\uparrow}$ for all x, we get the egalitarian criterion. Hence problem Π_ψ where ψ is a Choquet integral is NP-hard since $\Pi_{\min}$ is NP-hard (see Proposition 1). When used with an additive capacity, $C_v(x)$ boils down to a weighted sum (the utilitarian criterion). Of course when used with a non-additive capacity, function C_v offers additional descriptive possibilities. Among others we have the following nice property [6]:

Proposition 1 *If v is convex then $\forall x^1, \ldots, x^p \in \mathbb{R}^n$, $\forall k = 1, \ldots, p$, $\forall \lambda_1, \ldots, \lambda_p \geq 0$ such that $\sum_{i=1}^{p} \lambda_i = 1$ we have:*

$$C_v(x^1) = \ldots = C_v(x^p) \Rightarrow C_v(\sum_{i=1}^{p} \lambda_i x^i) \geq C_v(x^k) \quad (12)$$

Property (12) named "preference for diversification" in the context of portofolio management (see [6]) can be re-interpreted in terms of fairness because it means that smoothing or averaging a cost vector makes the society of agents better off. For example, let us consider a multiagent allocation problem with 2 agents and 3 different solutions with utility vectors $x^1 = (10, 20)$, $x^2 = (20, 10)$ and $x^3 = (12, 12)$. If $v(\{1\}) = v(\{2\}) = 0.1$ we have $C_v(x^1) = 10 + 0.1(20 - 10) = 12$; similarly $C_v(x^2) = C_v(x^3) = 12$. The average vector of $\{x^1, x^2, x^3\}$ is $\bar{x} = (14, 14)$ with $C_v(\bar{x}) = 14$, which is better than x^1, x^2 and x^3. This illustrates the impact of Property (12). Thus it seems interesting to study the maximization of a Choquet integral with a convex capacity v.

We now introduce a first linear reformulation of the problem of finding a C_v-optimal allocation. It is based on the following result due to Schmeidler [23] that holds for any convex capacity v:

$$C_v(x) = \min_{\lambda \in core(v)} \sum_{i=1}^{n} \lambda(\{i\}) x_i \quad (13)$$

Equation (13) suggests that $C_v(x)$ can also be seen as the optimal value of the following linear program:

$$(\mathcal{P}_{C_v}) \quad \begin{array}{l} \min \displaystyle\sum_{i=1}^{n} \lambda_i x_i \\ s.t. \left\{ \ v(A) \leq \sum_{i \in A} \lambda_i \quad \forall A \subseteq N \right. \\ \lambda_i \geq 0 \ \ i = 1 \ldots n \end{array}$$

$C_v(x)$ can also be seen as the optimal value of the dual program:

$$(\mathcal{D}_{C_v}) \quad \begin{array}{l} \max \displaystyle\sum_{A \subseteq N} v(A) d_A \\ s.t. \left\{ \ \sum_{A \subseteq N: i \in A} d_A \leq x_i \quad i = 1 \ldots n \right. \\ d_A \geq 0 \ \ \forall A \subseteq N \end{array}$$

We present now a 0-1 linear program obtained by combination of $(\mathcal{D}_{C_v})$ with our initial problem Π_ψ for $\psi = C_v$ in Equation (1):

$$(\Pi'_{C_v}) \quad \begin{array}{l} \max \displaystyle\sum_{A \subseteq N} v(A) d_A \\ s.t. \left\{ \begin{array}{ll} \sum_{A \subseteq N: i \in A} d_A \leq \sum_{j=1}^{m} u_{ij} z_{ij} & i = 1, \ldots, n \\ \alpha_i \leq \sum_{j=1}^{m} z_{ij} \leq \beta_i & i = 1, \ldots, n \\ \alpha'_j \leq \sum_{i=1}^{n} z_{ij} \leq \beta'_j & j = 1, \ldots, m \end{array} \right. \\ z_{ij} \in \{0, 1\} \ \ \forall i, \forall j \\ d_A \geq 0 \ \ \forall A \subseteq N \end{array}$$

This linear program has $nm + 2^n - 1$ variables including nm assignment variables z_{ij} and $2^n - 1$ variables d_A for every non-empty set $A \subseteq N$. This reformulation can be used for fair optimization problems involving very few agents but it will become quickly intractable by standard LP-solver as the number of agents increases. Fortunately, in practice, Choquet integrals are often used with $k-$additive capacities which restricts the number of parameters involved in the model while keeping good descriptive possibilities. However, there is no obvious way of using k-additivity of v to simplify problem Π'_{C_v}. To go one step further in this direction we propose rewriting $C_v(x)$ as a function of its Möbius transform which gives (see [14]):

$$C_v(x) = \sum_{A \subseteq N} m(A) \min_{i \in A} x_i \quad (14)$$

This formulation of $C_v(x)$ is easily linearizable provided that Möbius masses $m(A)$ are positive. Capacities whose Möbius masses are positive are well-know. For 2-additive measures, they coincide with convex capacities. For larger values of k they form a subclass of convex capacities called *belief functions* [22]. In the sequel we will assume that v is a belief function generated from positive Möbius masses. Note that the convexity of belief functions guarantees fairness of solutions through Property (12). Now, assuming that all Möbius masses are positive the search of the Choquet-optimal allocation can be expressed from Π_ψ and (14) as follows:

$$(\Pi''_{C_v}) \quad \begin{array}{l} \max \displaystyle\sum_{A \subseteq N} m(A) y_A \\ s.t. \left\{ \begin{array}{ll} y_A \leq \sum_{j=1}^{m} u_{ij} z_{ij} & \forall A \subseteq N, \forall i \in A \\ \alpha_i \leq \sum_{j=1}^{m} z_{ij} \leq \beta_i & i = 1, \ldots, n \\ \alpha'_j \leq \sum_{i=1}^{n} z_{ij} \leq \beta'_j & j = 1, \ldots, m \end{array} \right. \\ z_{ij} \in \{0, 1\} \ \ \forall i, \forall j \end{array}$$

where $y_A, A \subseteq N$ are auxiliary variables used to linearize min operations. This problem has as many variables as problem Π'_{C_v} but more constraints. Fortunately it can be significantly reduced in size under the $k-$additivity assumption. For example, with a 2-additive convex capacity, problem Π''_{C_v} has only $nm + (n^2 + n)/2$ variables including nm assignment variables and $(n^2 + n)/2$ variables y_A (one for each singleton and pair) and only $2(n + m) + n^2$ constraints.

5 Numerical tests

This section gives some numerical tests for the different problems presented before. We performed these tests using ILOG CPLEX 12.1

on a computer with 8 Gb of memory and an Intel Core 2 Duo 3.33 GHz processor. Table 1 gives the results obtained for the paper assignment problem modeled as follows: $n = m/4$, each reviewer receives at most 9 papers ($\alpha_i = 0$ and $\beta_i = 9$), a paper has to be reviewed by exactly 2 reviewers ($\alpha'_j = \beta'_j = 2$), and a reviewer expresses his preferences for reviewing a paper with a number between 0 and 5 (i.e. $u_{ij} \in [1, 5]$). Table 2 represents the results obtained for the Santa Claus problem with $n = m/4$ agents, $\alpha_i = 0, \beta_i = m$ and $\alpha'_i = \beta'_i = 1$ and the same utility functions. Computation times $t_{\min}, t_W, t_{C_v}$ expressed in seconds represent average solution times over 20 random instances of the same size m (number of objects) for problems $\Pi_{\min}, \Pi_W$ and Π''_{C_v} respectively. For the Gini social-evaluation function we used the classical instance given in Equation (7). For the Choquet integral, we used 2-additive convex capacities generated from randomly drawn positive Möbius masses.

m	$t_{\min}$
400	0.36
800	6.81
1200	21.25
1600	56.21
2000	85.54
2400	112.27
2800	181.75
3200	270.45
3600	496.93

m	t_W	t_{C_v}
200	1.29	0.04
300	2.77	1.36
400	7.51	3.81
500	18.34	8.79
600	36.19	18.83
700	68.39	52.34
800	120.61	89.37
900	177.65	165.14
1000	271.39	342.19

Table 1. Computation times (s) for the paper assignment problem

m	$t_{\min}$
400	0.56
800	8.52
1200	34.15
1600	95.82
2000	201.63
2400	329.29
2800	550.86
3200	794.26
3600	1169.84

m	t_W	t_{C_v}
200	0.49	0.08
300	6.91	3.59
400	24.42	9.78
500	77.91	27.75
600	154.25	55.47
700	359.54	133.55
800	518.94	181.91
900	979.08	393.77
1000	1547.87	646.11

Table 2. Computation times for (s) the Santa Claus problem.

6 Conclusion

We have discussed various criteria enabling to incorporate the idea of fairness in multiagent optimization problems. For each of these criteria we have provided a reformulation of the problem as a 0-1 linear program (see problems $\Pi_{\min}, \Pi_W, \Pi'_{C_v}$ and Π''_{C_v}) that gives the optimal solution. Numerical tests have shown that these linear programs are solvable with standard solvers for real size problems. Of course, the use of a general Choquet integral gets more heavy when the number of agents increases. Fortunately, the use of k-additive capacities with small k offers a good way of reducing the complexity of the model (number of parameters) and the size of the LP to be solved (Π''_{C_v}) while keeping very good descriptive possibilities compared to standard linear models.

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