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Refining the Notion of Effort

Bernhard Heinemann[1]

Abstract. The aim of this paper is to incorporate a qualitative approach to measurement into the more general concept of effort. The fundamental idea to achieve this goal is to assign suitable modalities to the methods of measurement being available to an agent. The knowledge of the agent changes, in fact, generally increases by applying such methods. Thus, the system should be able to describe this change of knowledge as well. We develop an appropriate logic, show that it satisfies some of the fundamental properties that are desirable at any rate, and discuss possible extensions.

1 INTRODUCTION

Modal logic provides a useful tool for qualitative modelling, as a wide field of application shows. Let us mention the notion of *knowledge* as an example substantiating this. Not only describing the knowledge of agents in a modal way has led to new insights into the nature of knowledge, but it enables one also to analyse multi-agent scenarios occurring in distributed systems, social procedures, or mathematical economics; see [2] for a detailed description regarding this. The basic postulate for corresponding *epistemic logics* is that knowledge is represented by an agent's *view of the world*. Such view consists of the states the agent considers possible, or, in other words, which are indistinguishable to the agent at the present point in time. Thus, this set of states may also be called an *epistemic state* of the agent.

The epistemic state usually changes in the course of time. If it shrinks, then the knowledge of the agent increases to the same extent, since certain alternatives to the actual state of the world are ruled out in some way. This is the case we are interested in in this paper. And we are particularly interested in the reasons for such shrinkage.

Generally speaking, the shrinkage of an agent's epistemic state indicates that some kind of *effort* is being expended, where the term 'effort' is said to embody the general idea of 'spending resources'. Thus, if it is irrelevant what kind of effort is involved, then this notion itself can be modelled by expressing *shrinking* suitably.

A modal logic dealing in such a way with both knowledge and effort was proposed in the paper [5]. We recall the underlying bi-modal language, $\mathcal{L}$, here. Basically, $\mathcal{L}$ comprises a modality K describing the knowledge of an agent under discussion, and a second one, $\Box$, measuring the effort to acquire knowledge qualitatively. The relevant semantic domains, called *subset spaces,* consist of a non-empty set X of states of the world, a set $\mathcal{O}$ of subsets of X representing the epistemic states of the agent, and a valuation V determining the states where the atomic propositions are true. $\mathcal{L}$-formulas are interpreted in subset spaces with respect to *neighbourhood situations* (x, U), for which $x \in U \in \mathcal{O}$ is valid by definition. The operator K quantifies over all states contained in some epistemic state $U \in \mathcal{O}$ then, whereas $\Box$ quantifies 'downward' over all $U' \in \mathcal{O}$ contained in U since shrinking sets of alternatives and gaining knowledge correspond to each other. In this way, the language $\mathcal{L}$ formalises our intuition regarding knowledge and effort.

We get on to the very topic of this paper now. While the idea of effort is general enough to subsume as diverse knowledge acquisition procedures as learning, computation, or measurement, we want to address, by way of example, the latter a bit more concrete. Suppose that various measurement methods (e.g., an optical and an electron microscope) are available to an agent (e.g., a molecular biologist), how to distinguish corresponding measuring results qualitatively by means of the formal model just sketched? As was indicated in the abstract already, we assign a modal operator to each such method. The scope of this operator is the set of all epistemic states the agent may take after an application of the respective method, hence a subset of $\mathcal{O}$. Of course, the set $\mathcal{O}$ need not necessarily be exhausted by all these subsets since not every epistemic state of the agent must be obtained by some measurement. Be that as it will, we equip the general notion of effort with some extra structure. A finite number of measurement methods are considered in addition here.

In the subsequent part of the paper, we therefore extend the language and the logic of knowledge and effort by finitely many further modalities. We proceed as follows. In Section 2, we define the language underlying the framework described above precisely. We then deal with an axiomatisation of the set of all valid formulas of this language. Our axiom system turns out to be sound and semantically complete. Moreover, our logic is even decidable. – Up to that point, no interaction of the measurement modalities is assumed. In Section 3, however, we discuss issues relating to this, which leads to the probably most interesting aspects of our approach.

2 A LOGIC FOR KNOWLEDGE, EFFORT, AND MEASUREMENT

We now present a multi-modal language, $\mathcal{L}'$, which meets the above requirements. We define the syntax first. Let $\mathsf{Prop} = \{p, q, \ldots\}$ be a denumerably infinite set of symbols called *proposition variables,* representing the basic facts about the states of the world. Furthermore, let n be the number of measurement methods being available to the agent in question. Then, the set Form of *formulas* over Prop is defined by the rule $\alpha \ ::= \ p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \mathsf{K}\alpha \mid \Box\alpha \mid \Box_i\alpha$, where $i = 1, \ldots, n$. Later on, some of the boolean connectives that are missing here are treated as abbreviations. The dual modal operators belonging to K, $\Box$ and $\Box_i$ are denoted by L, $\Diamond$ and $\Diamond_i$, respectively. In accordance with the introduction, K is called the *knowledge operator* and $\Box$ the *effort operator,* whereas the $\Box_i$ are called the *measurement operators* as of now ($i = 1, \ldots, n$).

Second, we fix the semantics of $\mathcal{L}'$. For a start, we define the relevant domains and comment on this definition afterwards. We let

[1] FernUniversität in Hagen, Germany, email: bernhard.heinemann@fernuni-hagen.de

$\mathcal{P}(X)$ designate the powerset of a given set X.

Definition 1 (Multi-subset spaces) *1. Let X be a non-empty set of states, and let $\mathcal{O}, \mathcal{O}_1, \ldots, \mathcal{O}_n \subseteq \mathcal{P}(X)$ be subsets of X such that $\mathcal{O}_i \subseteq \mathcal{O}$ for $i = 1, \ldots, n$. Then the triple $\mathcal{S} := (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n})$ is called a* multi-subset frame.

2. *Let $\mathcal{S} = (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n})$ be a multi-subset frame. The set $\mathcal{N}_{\mathcal{S}} := \{(x, U) \mid x \in U$ and $U \in \mathcal{O}\}$ is called the* set of neighbourhood situations *of $\mathcal{S}$.*
3. *Let $\mathcal{S} = (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n})$ be a multi-subset frame. An $\mathcal{S}$-*valuation *is a mapping $V : \mathsf{Prop} \to \mathcal{P}(X)$.*
4. *Let $\mathcal{S} = (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n})$ be a multi-subset frame and V an $\mathcal{S}$-valuation. Then, $\mathcal{M} := (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n}, V)$ is called a* multi-subset space *(*based on $\mathcal{S}$*).*

Note that $\mathcal{O}$ represents the set of *all* of the agent's epistemic states, whereas, for $i = 1, \ldots, n$, the set $\mathcal{O}_i$ consists of all epistemic states that are obtainable by the i-th measurement method. The term 'neighbourhood situation' is introduced just to denominate the semantic atoms of our language. Note that the first component of such a situation indicates the actual world while the second displays the uncertainty of the agent about it. – For a given multi-subset space $\mathcal{M}$, we now define the relation of satisfaction, $\models_{\mathcal{M}}$, between neighbourhood situations of the underlying frame and formulas from Form. We confine ourselves to the clause for the measurement operators.

Definition 2 (Satisfaction and validity) *Let a multi-subset space $\mathcal{M} = (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n}, V)$ based on $\mathcal{S} = (X, \mathcal{O}, \{\mathcal{O}_i\}_{1 \leq i \leq n})$ be given, and let $(x, U) \in \mathcal{N}_{\mathcal{S}}$ be a neighbourhood situation. Then $(x, U) \models_{\mathcal{M}} \Box_i \alpha \;:\iff\; \forall U' \in \mathcal{O}_i : (x \in U' \subseteq U \Rightarrow (x, U') \models_{\mathcal{M}} \alpha)$, where $i \in \{1, \ldots, n\}$ and $\alpha \in \mathsf{Form}$. Furthermore, a formula α is called* valid in *$\mathcal{M}$ iff $(x, U) \models_{\mathcal{M}} \alpha$ for all neighbourhood situations of $\mathcal{S}$.*

Thus the measurement operators quantify downward over certain elements of the set of subsets that is associated with these in each case. – We now turn to an axiom system capturing the measurement operators. For $i = 1, \ldots, n$, we take the following schemata.

1. $\Box_i(\alpha \to \beta) \to (\Box_i \alpha \to \Box_i \beta)$ 2. $\Box_i \alpha \to \Box\Box_i \alpha$
3. $\mathsf{K}\Box_i \alpha \to \Box_i \mathsf{K} \alpha$ 4. $\Box \alpha \to \Box_i \alpha$,

where $\alpha, \beta \in \mathsf{Form}$. This means that we have *distribution,* 'near-transitivity' of the accessibility relation $\stackrel{\Box_i}{\to}$ associated with $\Box_i$, the *Cross Axiom* for K and $\Box_i$ (c.f. [5]), and *inclusion,* i.e., the accessibility relation $\stackrel{\Box_i}{\to}$ is contained in the one belonging to $\Box$.

The *logic of multi-subset spaces,* LMS, is the poly-modal logic determined by the just listed axioms, the axioms for knowledge and effort from [5], and the standard modal proof rules *modus ponens* and *necessitation* (with respect to each single modality). Now, the first of our main results states the soundness and completeness of this logic.

Theorem 3 (Soundness and completeness) *A formula $\alpha \in \mathsf{Form}$ is valid in all multi-subset spaces iff $\alpha \in \mathsf{LMS}$.*

While the proof of the completeness part of Theorem 3 requires an infinite step-by-step construction of the model falsifying a given non-derivable formula, a rather involved *filtration* (c.f. [3], § 4) argument is used for the proof of the subsequent decidability result. Prior to this, additional difficulties arising from the fact that the *finite model property* (c.f. [1], § 6.2) is invalid for LMS have to be circumvented.

Theorem 4 (Decidability) *The logic LMS is decidable.*

3 INTERDEPENDENCE OF MEASUREMENTS

The logical system developed so far is insensitive to the mutual relationship of the measurement methods, which often exists in reality though. For example, an electron microscope generally reveals more precise measuring results than an optical one so that the entirety of such results should in a sense be 'of higher quality' than that arising from the optical instrument. This should be reflected in an agent's epistemic states accruing from that in each case. Thus, the question comes up whether circumstances like those of this example can be described adequately by means of our formal model. Subsequently, we discuss a few issues relating to this by way of illustration.

First, let $n = 2$, and let $\Box_o$ and $\Box_e$ correspond to the devices from the example in a way which suggests itself. Furthermore, let $\top$ be a formula which is true everywhere. Then the formula (1) $\Box_o \Diamond_e \top$ exactly says that every knowledge state arising from an optical measurement contains a knowledge state arising from an electron-optical one; i.e., the agent might know more in the second case. Thus, formula (1) captures the idea of comparing the quality of measurement methods in this special case.

We now abstract from the example and let n be an arbitrary natural number again. Our next aim is to compare measurement methods with regard to the relation 'leads to fewer results'. Let us assume for the moment that one of the measuring systems being available to the agent, say j, produces a *smallest* set $\mathcal{O}_j$ of epistemic states of the agent. Then this situation can be expressed by the following schema (2) $\Box_i \alpha \to \Box_j \alpha$, for all $i \in \{1, \ldots, n\}$. The method j may be called the *most specialised* one, as it is diametrically opposed to the general effort operator.

The third schema we are interested in is (3) $\Box_i \alpha \to \Box_j \alpha$ with $i \leq j$, where $i, j \in \{1, \ldots, n\}$. A *linear ordering* is imposed on the measurement methods in this way. Here, too, we have a most specialised method, n, and the degree of specialisation of any method i is determined by its occurrence in the linear ordering.

The case that all methods are *equivalent* also deserves attention. The term 'being equivalent' means 'leading to the same results' here. Thus the schema (4) $\Box_i \alpha \leftrightarrow \Box_j \alpha$, for all $i, j \in \{1, \ldots, n\}$, is appropriate for this case.

Considering the latter schemata (and others, which must be omitted here) is inspired by [4], where these formulas specify certain properties of knowledge sharing in groups of agents. We now obtain the following theorem for $\mathsf{LMS}_i := \mathsf{LMS} + (i)$, where $i = 1, \ldots, 4$.

Theorem 5 (Extended completeness and decidability) *For all $i \in \{1, \ldots, 4\}$, the logic LMS_i is sound and complete with respect to the class of all multi-subset spaces satisfying the corresponding property. Moreover, LMS_i is decidable.*

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