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Horn Belief Change: A Contraction Core

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Abstract. We show that Booth *et al.*'s Horn contraction based on infra-remainder sets corresponds exactly to kernel contraction for belief sets. This result is obtained via a detour through Horn contraction for belief bases, which supports the conjecture that Horn belief change is best viewed as a "hybrid" version of belief set change and belief base change. Moreover, the link with base contraction gives us a more elegant representation result for Horn contraction for belief sets in which a version of the Core-retainment postulate features.

1 INTRODUCTION

While there has been some work on *revision* for Horn clauses [4, 7, 6], it is only recently that attention has been paid to its *contraction* counterpart. Delgrande [3] investigated two classes of contraction functions for Horn *belief sets*, viz. e-contraction and i-contraction, while Booth *et al.* [2] subsequently extended Delgrande's work.

A *Horn clause* has the form $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$ with $n \geq 0$, and p_i, q atoms. A *Horn sentence* is a conjunction of Horn clauses. The semantics is the same as for propositional logic, just restricted to Horn sentences. A *Horn base* B is a set of Horn sentences. We use $Cn_{\mathsf{HL}}(X)$ to denote the set of all consequences of X which are in the language of Horn sentences. A *Horn belief set* H is a Horn base closed under logical consequence (containing only Horn sentences).

Delgrande's construction method for Horn contraction is in terms of partial meet contraction [1]. The standard definitions of remainder sets, selection functions, partial meet contraction, as well as maxichoice and full meet contraction all carry over for the Horn case. We refer to these as e-remainder sets (denoted by $H \bot_e \varphi$), e-selection functions, partial meet e-contraction, maxichoice e-contraction and full meet e-contraction respectively. As in the full propositional case, all e-remainder sets are also Horn belief sets, and all partial meet e-contractions (and therefore the maxichoice e-contractions, as well as full meet e-contraction) produce Horn belief sets.

Booth *et al.* [2] argue that although all partial meet e-contractions are appropriate choices, they do not make up the set of *all* appropriate e-contractions. The argument is based on the observation that the convexity result for full propositional logic [2, Proposition 2.1] does not hold for Horn logic.

Example 1 *Let $H = Cn_{\mathsf{HL}}(\{p \rightarrow q, q \rightarrow r\})$. Then, for the e-contraction of H with $p \rightarrow r$, maxichoice yields either $H^1_{mc} = Cn_{\mathsf{HL}}(\{p \rightarrow q\})$ or $H^2_{mc} = Cn_{\mathsf{HL}}(\{q \rightarrow r, p \wedge r \rightarrow q\})$. Full meet yields $H_{fm} = Cn_{\mathsf{HL}}(\{p \wedge r \rightarrow q\})$. These are the only three partial meet e-contractions. Now consider the Horn belief set $H' = Cn_{\mathsf{HL}}(\{p \wedge q \rightarrow r, p \wedge r \rightarrow q\})$. It is clear that $H_{fm} \subseteq H' \subseteq H^2_{mc}$, but there is no partial meet e-contraction yielding H'.*

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In order to rectify this situation, Booth *et al.* [2] propose that *every* Horn belief set between full meet and some maxichoice e-contraction ought to be seen as an appropriate candidate for e-contraction: For Horn belief sets H and H', $H' \in H \downarrow_e \varphi$ iff there is some $H'' \in H \bot_e \varphi$ s.t. $(\bigcap H \bot_e \varphi) \subseteq H' \subseteq H''$. $H \downarrow_e \varphi$ is the set of *infra e-remainder sets* of H w.r.t. φ.

The intersection of any set of infra e-remainder sets is also an infra e-remainder set, and the set of infra e-remainder sets contains *all* Horn belief sets between some e-remainder set and the intersection of all e-remainder sets. This explains why e-contraction below is not defined as the intersection of infra e-remainder sets.

Definition 1 (Horn e-Contraction) *For H a Horn belief set, an infra e-selection function τ is a (partial) function from $\mathscr{P}(\mathscr{P}(\mathcal{L}_{\mathsf{H}}))$ to $\mathscr{P}(\mathcal{L}_{\mathsf{H}})$ s.t. $\tau(H \downarrow_e \varphi) = H$ when $H \downarrow_e \varphi = \emptyset$, and $\tau(H \downarrow_e \varphi) \in H \downarrow_e \varphi$ otherwise. An e-contraction $-_\tau$ is an infra e-contraction iff $H -_\tau \varphi = \tau(H \downarrow_e \varphi)$.*

Booth *et al.* show that infra e-contraction is captured by the six AGM postulates for belief set contraction, namely Closure, Success, Vacuity, Inclusion, and Extensionality, with Recovery replaced by the (weaker) postulate $(H -_e 6)$ below, and the Failure postulate.

$(H -_e 6)$ If $\psi \in H \setminus (H - \varphi)$, then there exists an X s.t. $\bigcap(H \bot_e \varphi) \subseteq X \subseteq H$ and $X \not\models \varphi$, but $X \cup \{\psi\} \models \varphi$

$(H -_e 7)$ If $\models \varphi$, then $H -_e \varphi = H$ (Failure)

$(H -_e 6)$ resembles the Relevance postulate for base contraction. Also, it is an unusual postulate since it refers to e-remainder sets. We shall give a more elegant characterisation of infra e-contraction. Before we do so, we first take a detour through base contraction.

2 A DETOUR VIA BASE CONTRACTION

For a base B, $X \in B \perp\!\!\!\perp \varphi$ if and only if (i) $X \subseteq B$; (ii) $X \models \varphi$; and (iii) for every X' s.t. $X' \subset X$, $X' \not\models \varphi$. $B \perp\!\!\!\perp \varphi$ is called the *kernel set* of B w.r.t. φ and the elements of $B \perp\!\!\!\perp \varphi$ are called the *φ-kernels of B*.

The result of a base kernel contraction is obtained by removing at least one element from every (non-empty) φ-kernel of B, using an *incision function*, which maps the kernel sets of B to a base. Given an incision function σ for a base B, the *base kernel contraction* $-_\sigma$ for B generated by σ is defined as: $B -_\sigma \varphi = B \setminus \sigma(B \perp\!\!\!\perp \varphi)$.

Base kernel contraction can be characterised by the same postulates for partial meet contraction, except that Relevance is replaced by the Core-retainment postulate below [5]:

$(B - 5)$ If $\psi \in (B \setminus (B - \varphi))$, then there is some $B' \subseteq B$ such that $B' \not\models \varphi$ but $B' \cup \{\psi\} \models \varphi$ (Core-retainment)

Definition 2 (Base Infra Remainder Sets) *For bases B and B', $B' \in B{\downarrow}\varphi$ iff there is some $B'' \in B{\perp}\varphi$ s.t. $(\bigcap B{\perp}\varphi) \subseteq B' \subseteq B''$. $B{\downarrow}\varphi$ is the set of* base infra remainder sets *of B w.r.t. φ.*

The definition of base infra remainder sets differs from that of infra e-remainder sets only in that (*i*) it deals with belief bases; and (*ii*) it is defined in terms of remainder sets for bases, and not for (Horn) belief sets.

Base infra remainder sets can be used to define a form of base contraction in a way that is similar to Definition 1.

Definition 3 (Base Infra Contraction) *A* base infra selection function *τ is a (partial) function mapping sets of Horn bases to a Horn base. A base contraction $-_\tau$ defined as $B -_\tau \varphi = \tau(B{\downarrow}\varphi)$ is a* base infra contraction*.*

A natural question to ask is how base infra contraction compares with base partial meet contraction and base kernel contraction. The following result, which plays a central role in this work, shows that base infra contraction corresponds exactly to base kernel contraction.

Theorem 1 *A base contraction for a base B is a base kernel contraction for B if and only if it is a base infra contraction for B.*

It is known that base kernel contraction is more general than base partial meet contraction — every base partial meet contraction is also a kernel contraction, but the converse does not hold. From Theorem 1 it thus follows that base infra contraction is more general than base partial meet contraction. This is not surprising, given that a similar result holds for partial meet e-contraction and infra e-contraction as we have seen in Example 1.

Theorem 1 has a number of other interesting consequences as well. On a philosophical note, it provides corroborative evidence for the contention that the kernel contraction approach is more appropriate than the partial meet approach. Theorem 1 adds to this by showing that a seemingly different approach to contraction (infra contraction), which is also at least as general as partial meet contraction for both base and belief set contraction, turns out to be identical to kernel contraction. As we shall see in the next section, Theorem 1 is also instrumental in "lifting" this result to the level of Horn belief sets.

3 AN EQUIVALENCE RESULT

Through the work of Booth *et al.* [2] we have already encountered partial meet contraction and infra contraction for Horn belief sets (partial meet e-contraction and infra e-contraction), but we have not yet defined a suitable version of kernel contraction for this case.

Definition 4 *Given a Horn belief set H and an incision function σ for H, the* Horn kernel e-contraction for H*, abbreviated as the* kernel e-contraction for H *is defined as $H \approx^e_\sigma \varphi = Cn_{\mathsf{HL}}(H -_\sigma \varphi)$, where $-_\sigma$ is the base kernel contraction for φ obtained from σ.*

Given the results on how kernel contraction, partial meet contraction and infra contraction compare for the base case (kernel contraction and infra contraction are identical, while both are more general than partial meet contraction), one would expect similar results to hold for Horn belief sets. And this is indeed the case. Firstly, infra e-contraction and kernel e-contraction coincide.

Theorem 2 *An e-contraction for a Horn belief set H is an infra e-contraction for H if and only if it is a kernel e-contraction for H.*

From Theorem 2 and Example 1 it follows that partial meet e-contraction is more restrictive than kernel e-contraction. When it comes to Horn belief sets, we therefore have exactly the same pattern as we have for belief bases — kernel contraction and infra contraction coincide, while both are strictly more permissive than partial meet contraction.

One conclusion to be drawn from this is that the restriction to the Horn case produces a curious hybrid between belief sets and belief bases for full propositional logic. On the one hand, Horn contraction deals with sets that are logically closed. But on the other hand, the results for Horn logic obtained in terms of construction methods are close to those obtained for belief base contraction.

4 A MORE ELEGANT CHARACTERISATION

In the end of Section 1 we remarked that the characterisation of infra e-contraction is somewhat unusual in that it refers directly to the construction method. Now we show that it is possible to provide a more elegant characterisation of infra e-contraction.

Theorem 3 *Every infra e-contraction satisfies the basic AGM postulates Closure, Vacuity, Success, Inclusion, Extensionality, together with Core-retainment. Conversely, every e-contraction which satisfies Closure, Vacuity, Success, Inclusion, Extensionality and Core-retainment is an infra e-contraction.*

This result was inspired by Theorem 1 which shows that base kernel and base infra contraction coincide. Given that Core-retainment is used in characterising base kernel contraction, Theorem 1 shows that there is a link between Core-retainment and base infra contraction, and raises the question of whether there is a link between Core-retainment and infra e-contraction. The answer, as we have seen in Theorem 3, is yes. This result provides more evidence for the hybrid nature of contraction for Horn belief sets — in this case the connection with base contraction is strengthened.

5 CONCLUDING REMARKS

In bringing Hansson's kernel contraction into the picture, we have made meaningful contributions to the investigation into contraction for Horn logic. The main contributions of this work are (*i*) a result which shows that infra contraction and kernel contraction for the base case coincide; (*ii*) lifting the previous results to Horn belief sets to show that infra contraction and kernel contraction for Horn belief sets coincide; and (*iii*) using these results as a guide to the provision of a more elegant characterisation of the representation result by Booth *et al.* for infra contraction as applied to Horn belief sets.

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