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Taking the Final Step to a Full Dichotomy of the Possible Winner Problem in Pure Scoring Rules[1]

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Abstract. The POSSIBLE WINNER problem asks, given an election where the voters' preferences over the candidates are specified only partially, whether a designated candidate can be made win. Betzler and Dorn [1] proved a result that is only one step away from a full dichotomy of this problem for the important class of pure scoring rules in the case of unweighted voters and an unbounded number of candidates: POSSIBLE WINNER is NP-complete for all pure scoring rules except plurality, veto, and the scoring rule with vector $(2,1,\dots,1,0)$, but is solvable in polynomial time for plurality and veto. We take the final step to a full dichotomy by showing that POSSIBLE WINNER is NP-complete also for the scoring rule with vector $(2,1,\dots,1,0)$.

1 Introduction

The computational complexity of problems related to voting systems is a field of intense study (see, e.g., [3]). The voters are commonly assumed to provide their preferences over the candidates via complete linear orderings of all candidates. However, in many real-life settings, this is often not the case: Some voters may have preferences over only some candidates, or it may happen that new candidates are introduced to an election after some voters have already cast their votes. It thus seems reasonable to assume only partial preferences from the voters. Konzcak and Lang [5] were the first to study voting with partial preferences, and they proposed the POSSIBLE WINNER problem that (for any given election system) asks, given an election with only partial preferences and a designated candidate c, whether c is a possible winner in some extension of the partial votes to linear ones. This problem was studied later on by other authors (see, e.g., [6, 1]). In particular, Betzler and Dorn [1] established a result that is only one step away from a full dichotomy result[3] of the POSSIBLE WINNER problem for the important class of pure scoring rules (for unweighted voters and any number of candidates). They showed NP-completeness for all but three pure scoring rules, namely plurality, veto, and the scoring rule with scoring vector $(2,1,\dots,1,0)$. For plurality and veto they showed that this problem is polynomial-time solvable, but the complexity of POSSIBLE WINNER for the scoring rule $(2,1,\dots,1,0)$ was left open. Taking the final step to a full dichotomy result, we show that POSSIBLE WINNER is NP-complete also for the scoring rule with vector $(2,1,\dots,1,0)$.

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[3] Dichotomy results are particularly important, as they completely settle the complexity of a whole class of related problems by providing an easy-to-check condition that tells the hard cases apart from the easily solvable cases. For example, Hemaspaandra and Hemaspaandra [4] provided a dichotomy result for the manipulation problem in scoring rules with weighted voters.

2 Definitions and Notation

An election (C,V) is specified by a set $C=\{c_1,c_2,\dots,c_m\}$ of candidates and a list $V=(v_1,v_2,\dots,v_n)$ of votes over C. In the most common model of representing preferences, each such vote is a linear order[4] of the form $c_{i_1} > c_{i_2} > \cdots > c_{i_m}$ where $\{i_1,i_2,\dots,i_m\} = \{1,2,\dots,m\}$, and $c_{i_k} > c_{i_\ell}$ means that candidate c_{i_k} is (strictly) preferred to candidate c_{i_ℓ}. A voting system is a rule to determine an election's winner(s). Scoring rules (a.k.a. scoring protocols) are an important class of voting systems. Every scoring rule with m candidates is specified by a scoring vector $\vec{\alpha}=(\alpha_1,\alpha_2,\dots,\alpha_m)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$, where each α_j is a nonnegative integer. For an election (C,V), each voter $v \in V$ gives a_j points to the candidate ranked at jth position in his or her vote. Summing up all points a candidate $c \in C$ receives from all voters in V, we obtain $score_{(C,V)}(c)$, c's score in (C,V). Whoever has the highest score wins the election. If there is only one such candidate, he or she is the unique winner. Betzler and Dorn [1] focused on so-called pure scoring rules. A scoring rule is *pure* if for each $m \geq 2$, the scoring vector for m candidates can be obtained from the scoring vector for $m-1$ candidates by inserting one additional score value at any position subject to satisfying $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m$. We will study only the pure scoring rule that for $m \geq 2$ candidates is defined by the scoring vector $(2,1,\dots,1,0)$: The first candidate gets two points, the last candidate gets zero points, and the $m-2$ other candidates get one point each. We thus distinguish between the first, a middle, and the last position in any vote.

The POSSIBLE WINNER problem is defined for partial rather than linear votes. For a set C of candidates, a *partial vote over* C is a transitive, antisymmetric (not necessarily total) relation on C. For any two candidates c and d in a partial vote, we write $c \succ d$ if c is (strictly) preferred to d. If a candidate c is preferred to each candidate from a set D of candidates, we write $c \succ D$ to mean $c \succ d$ for all $d \in D$. A linear vote v' over C *extends* a partial vote v over C if $v \subseteq v'$, i.e., for all $c,d \in C$, if $c \succ d$ in v then $c > d$ in v'. A list $V'=(v'_1,v'_2,\dots,v'_n)$ of linear votes is an extension of a list $V=(v_1,v_2,\dots,v_n)$ of partial votes if for each i, $1 \leq i \leq n$, $v'_i \in V'$ extends $v_i \in V$.

For a voting system $\mathscr{E}$, the POSSIBLE WINNER problem is defined as follows: Given a set C of candidates, a list V of partial votes over C, and a designated candidate $c \in C$, is there an extension V' of V to linear votes over C such that c is a winner of election (C,V') under voting system $\mathscr{E}$? (This defines the problem in the nonunique-winner case; for its unique-winner variant, simply replace "a winner" by "the unique winner.") Due to space we focus on the nonunique-winner case; the unique-winner case can be handled as described in [1].

[4] A *linear order* L *on* C is a total, transitive, and antisymmetric relation on C, i.e., (i) for any two distinct $c,d \in C$, either cLd or dLc; (ii) for all $c,d,e \in C$, if cLd and dLe then cLe; and (iii) for all $c,d \in C$, if cLd then dLc does not hold. Note that antisymmetry of L implies irreflexivity of L.

3 The Final Step to a Full Dichotomy Result

Our proof of Theorem 2 below uses the notion of maximum partial score defined in [1]. Let C be a set of candidates, $c \in C$, and let $V = V^\ell \cup V^p$ be a list of votes over C, where V^ℓ contains only linear votes and V^p contains partial votes such that c's score is fixed, i.e., the exact number of points c receives from any $v \in V^p$ is known, no matter to which linear vote v is extended. For each $d \in C - \{c\}$, define the *maximum partial score of d with respect to c* (denoted by $s_p^{\max}(d,c)$) to be the maximum number of points that d may get from (extending to linear votes) the partial votes in V^p without defeating c in (C,V') for any extension V' of V to linear votes. Note that for any such V', $s_p^{\max}(d,c) = score_{(C,V')}(c) - score_{(C,V^\ell)}(d)$. The following lemma will be helpful.

Lemma 1 (Betzler and Dorn [1]) *Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be any scoring rule, let C be a set of $m \geq 2$ candidates with designated candidate $c \in C$, and let V^p be a list of partial votes in which the score of c is fixed. Suppose that the following two properties hold: (1) There is a candidate $d \in C - \{c\}$ such that $s_p^{\max}(d,c) \geq \alpha_1 |V^p|$. (2) For each $c' \in C - \{c\}$, the maximum partial score of c' with respect to c can be written as a linear combination of the score values, $s_p^{\max}(c',c) = \sum_{j=1}^m n_j \alpha_j$, with $m = |C|$, $n_j \in \mathbb{N}$, and $\sum_{j=1}^m n_j \leq |V^p|$.*

Then a list V^ℓ of linear votes can be constructed in polynomial time such that for all $c' \in C - \{c\}$, $score_{(C,V^\ell)}(c') = score_{(C,V')}(c) - s_p^{\max}(c',c)$, where V' is an extension of V^p to linear votes.

Theorem 2 POSSIBLE WINNER *(both in the nonunique-winner case and in the unique-winner case) is* NP*-complete for the pure scoring rule with scoring vector* $(2,1,\ldots,1,0)$.

Proof. Membership in NP is obvious. Our NP-hardness proof is by a reduction from the NP-complete HITTING SET problem which is defined as follows: Given a finite set X, a collection $\mathscr{S} = \{S_1, \ldots, S_n\}$ of nonempty subsets of X (i.e., $\emptyset \neq S_i \subseteq X$ for each i, $1 \leq i \leq n$), and a positive integer k, is there a subset $X' \subseteq X$ with $|X'| \leq k$ such that X' contains at least one element from each subset in $\mathscr{S}$?

Let $(X, \mathscr{S}, k)$ be a given HITTING SET instance with $X = \{e_1, e_2, \ldots, e_m\}$ and $|\mathscr{S}| = n$. From $(X, \mathscr{S}, k)$ we construct a POSSIBLE WINNER instance with candidate set $C = \{c,h\} \cup \{x_i, x_i^1, \ldots, x_i^n, y_i^1, \ldots, y_i^n, z_i^1, \ldots, z_i^n \mid 1 \leq i \leq m\}$ and designated candidate c. The list of votes $V = V^\ell \cup V^p$ consists of a list V^ℓ of linear votes and a list V^p of partial votes. V^p consists of three sublists, V_1^p, V_2^p, and V_3^p. In particular, V_1^p contains k votes of the form $h \succ C - \{h, x_1, \ldots, x_m\} \succ \{x_1, \ldots, x_m\}$, V_2^p contains the following $2n+1$ votes for each i, $1 \leq i \leq m$:

$$
\begin{array}{lll}
v_i: & h \succ C - \{h, x_i, y_i^1\} \succ \{x_i, y_i^1\}, & \\
v_i^j: & y_i^j \succ C - \{y_i^j, z_i^j, h\} \succ h & \text{for } 1 \leq j \leq n, \\
w_i^j: & x_i^j \succ C - \{x_i^j, y_i^{j+1}, z_i^j\} \succ y_i^{j+1} & \text{for } 1 \leq j \leq n-1, \\
w_i^n: & x_i^n \succ C - \{x_i^n, z_i^n, h\} \succ h, &
\end{array}
$$

and V_3^p contains for each i, $1 \leq i \leq n$, the vote $T_i \succ C - \{T_i, h\} \succ h$, where T_i contains the candidates x_j^i, $j \in \{\ell \mid e_\ell \in S_i\}$, corresponding to the elements in S_i, $1 \leq i \leq n$. For each i, $1 \leq i \leq m$, and j, $1 \leq j \leq n$, the maximum partial scores are: $s_p^{\max}(x_i) = |V^p| - 1$, $s_p^{\max}(x_i^j) = |V^p| + 1$, $s_p^{\max}(y_i^j) = s_p^{\max}(z_i^j) = |V^p|$, $s_p^{\max}(h) \geq 2|V^p|$. By Lemma 1, V^ℓ ensures that all candidates other than c can get only their maximum partial scores with respect to c in the partial votes.

If $\mathscr{S}$ has a hitting set of size k then c is a possible winner in (C,V), via extending V^p to linear votes as follows:

		$e_i \in X'$	$e_i \notin X'$
V_1^p :		$h > \cdots > x_i$	
V_2^p :	v_i :	$h > \cdots > x_i > y_i^1$	$h > \cdots > y_i^1 > x_i$
	$v_i^j, 1 \leq j \leq n$:	$y_i^j > \cdots > z_i^j$	$z_i^j > y_i^j > \cdots > h$
	$w_i^j, 1 \leq j < n$:	$z_i^j > x_i^j > \cdots > y_i^{j+1}$	$x_i^j > \cdots > y_i^{j+1} > z_i^j$
	w_i^n :	$z_i^n > x_i^n > \cdots > h$	$x_i^n > \cdots > h > z_i^n$
V_3^p :		$x_j^i > \cdots > h$ for some $j \in \{\ell \mid e_\ell \in S_i\}$	

Conversely, assume that c is a possible winner for (C,V). Then no candidate may get more points in V^p than his or her maximum partial score with respect to c. Since at most k different x_i may take a last position in V_1^p, at least $n-k$ different x_i must take a last position in v_i. We will now show that for those x_i it is not possible that a candidate x_i^j takes a first position in any vote of V_3^p. Fix any such x_i. Since x_i takes the last position in v_i, y_i^1 takes a middle position in this vote and gets one point. The only vote in which the score of y_i^1 is not fixed is v_i^1. Without the points from this vote, y_i^1 already gets $|V^p| - 1$ points, so y_i^1 cannot get two points in v_i^1, and z_i^1 takes the first position in v_i^1. Without the points from w_i^1, z_i^1 gets $|V^p|$ points and must take the last position in w_i^1. The first position in w_i^1 is then taken by x_i^1, so x_i^1 cannot take a first position in a vote from V_3^p. Candidate y_i^2 gets one point in w_i^1, and by a similar argument as above, x_i^2 is placed at the first position in w_i^2. Repeating this argument, we have that for each j, $1 \leq j \leq n$, x_i^j is placed at the first position in w_i^j and thus cannot take a first position in a vote from V_3^p. This means that all first positions in the votes of V_3^p must be taken by those x_i^j for which x_i takes the last position in a vote from V_1^p. This is possible only if the x_i^j are not at the first position in w_i^j. Thus z_i^j must take this position. Due to the maximum partial scores of these candidates, this is possible only if z_i^j takes the last position in v_i^j. Then y_i^j takes the first position in this vote. This is possible, since y_i^j can take a middle position in v_i for $j=1$, and in v_i^j for $2 \leq j \leq n$. Hence all x_i^j, where x_i takes the last position in the votes of V_1^p, may take the first position in the votes of V_3^p, and the e_i corresponding to those x_i must form a hitting set of size at most k for $\mathscr{S}$. ❑

Elkind et al. [2] introduced the notion of swap bribery. Here, an external agent seeks to make a distinguished candidate c win the election by bribing some voters to swap two consecutive candidates in their preference order. Since POSSIBLE WINNER is a special case of their swap bribery problem, Theorem 2 implies that the swap bribery problem for the scoring rule $(2,1,\ldots,1,0)$ is NP-hard as well.

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