A stress-dilatancy relation for cemented sands

Une relation de tension-dilatancy pour les sables cimentés

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ABSTRACT
In numerical modeling of uncemented soils, Rowe's stress-dilatancy relation for frictional materials was widely used and supported by a large body of experimental results. Even though the derivation of the theory was questioned because the principle of minimum energy no longer applies when friction is involved, De Josselin de Jong (1976) did prove, with an alternative approach based on the laws of friction, that Rowe's final conclusions and his stress-dilatancy relation were valid. Besides frictional materials, Rowe (1962) also proposed a stress-dilatancy relation for cohesive-frictional materials. Recently, researchers applied this relation for cemented sands. Although the Rowe's relation for frictional materials was proved to be correct, the validity of his stress-dilatancy relation for cohesive-frictional materials was never checked. This must be done before the application of that relation to model cohesive-frictional materials. This paper will show that the Rowe's stress-dilatancy relation for cohesive-frictional materials is not correct. A correct stress-dilatancy relation for cemented sands will be proposed, which was derived using the friction laws used by De Josselin de Jong (1976).

RÉSUMÉ
Dans le fait de modeler numérique de sols non cimentés, la relation de tension-dilatancy de Rowe pour le matériel à friction a été largement utilisée et soutenue par un grand corps de résultats expérimentaux. Bien que la dérivation de la théorie ait été questionnée parce que le principe d'énergie minimale ne fait plus une demande quand la friction est impliquée, De Josselin de Jong (1976) s'est vraiment avéré, avec une approche alternative basée sur les lois de friction, que les conclusions finales de Rowe et sa relation de tension-dilatancy étaient valides. En plus du matériel à friction, Rowe (1962) a aussi proposé une relation de tension-dilatancy pour le matériel à-friction-cohésif. Récemment, les chercheurs ont appliqué cette relation pour les sables cimentés. Bien que la relation du Rowe pour le matériel à friction ait été prouvée pour être correcte, la validité de sa relation de tension-dilatancy pour le matériel à-friction-cohésif n'a jamais été vérifiée. Cela doit être fait avant l'application de cette relation pour modeler le matériel à-friction-cohésif. Ce papier montrera que la relation de tension-dilatancy du Rowe pour le matériel à-friction-cohésif n'est pas correcte. Une relation de tension-dilatancy correcte pour les sables cimentés sera proposée, qui a été tiré en utilisant les lois de friction utilisées par De Josselin de Jong (1976).

Keywords : stress-dilatancy relation; flow rule; stress transformation; friction angle; dilation angle; cohesion

1 INTRODUCTION

In recent years, the study of cemented sands has received an increasing amount of attention in the field of geotechnical engineering. Comparing with uncemented sands, cemented sands have cementation bonds between the soil particles imparting on the soil a true cohesive strength component. Thus, the strength of cemented sands is a combination of cohesion (between soils particles), dilatancy (which develops under shearing) and friction (at the particle contacts). The behavior of cemented sands is more complicated than uncemented sands because of the existence of cementation between particles. Analyzing and modeling the mechanical behavior of cemented sands becomes an interesting and challenging topic.

Rowe (1962, 1972) proposed a stress-dilatancy relation which has been widely used in simulating the stress-strain behavior of uncemented sands and other granular materials (Hughes et al. 1977; Bolton 1986; Jefferies 1993). By applying the principle of energy minimization, Rowe (1962, 1972) showed that:

$$\frac{\sigma_1}{\sigma_3\left(1-\frac{d\varepsilon_v}{d\varepsilon_1}\right)}=\tan^2\left(\frac{\pi}{4}+\frac{\phi_c}{2}\right) \tag{1}$$

where σ_1 and σ_3 are the major and minor principal effective stresses, respectively (note that in this paper we deal exclusively with effective stresses and the customary primes for effective stresses will be omitted); $d\varepsilon_1$ and $d\varepsilon_v$ are the major principal strain increment and volumetric strain increment, respectively; and ϕ_c is the critical-state friction angle, a material constant. Rowe's stress-dilatancy relation for purely frictional materials , such as uncemented sand, was supported by many experimental results. However, the derivation of the Rowe's relation was questioned because the minimum energy principle will be violated if friction is involved in the system. Despite questioning Rowe's derivation, De Josselin de Jong (1976) used an alternative approach based on the laws of friction and proved that Rowe's stress-dilatancy relationship was valid.

In addition, Rowe also provided a stress-dilatancy relation for cohesive-frictional materials (e.g., cemented sands) based on the principle of energy minimization (Rowe 1962; Rowe et al. 1963):

$$\frac{\sigma_1}{\sigma_3\left(1-\frac{d\varepsilon_v}{d\varepsilon_1}\right)}=\tan^2\left(\frac{\pi}{4}+\frac{\phi_c}{2}\right)+\frac{2c}{\sigma_3}\tan\left(\frac{\pi}{4}+\frac{\phi_c}{2}\right) \tag{2}$$

where c is the interparticle cohesion. Eq. (1) is just a special case of Eq. (2), resulting from making the cohesion term c equal

Proceedings of the 17th International Conference on Soil Mechanics and Geotechnical Engineering
M. Hamza et al. (Eds.)

doi:10.3233/978-1-60750-031-5-510

to zero. Eq. (2) can thus be considered as the generalized Rowe's stress-dilatancy relation, which was intended for application to both frictional and cohesive-frictional materials.

In geotechnical engineering, typical cohesive-frictional materials include natural soils, stabilized soils, and rocks. One of the important properties of these materials is that there exist cementation bonds between particles, and the contribution of these bonds to shear strength may be represented by interparticle cohesion. When external forces are applied on such a material, the input energy will be used to change the volume of the material, to overcome interparticle friction, and to degrade the cement bonds between particles. Harberfield (1997) suggested the use of the Rowe (1962) stress-dilatancy relation when studying the effects of cracking in soft rocks during the pressuremeter test. Later, Cecconi et al. (2001) analyzed the experimental data on a pyroclastic weak rock by using Eq. (2), Mántaras & Schnaid (2002) also used Eq. (2) in cavity expansion analysis in dilatant cohesive-frictional materials.

As mentioned earlier, Rowe's stress-dilatancy relation was derived based on the incorrect assumption that energy minimization would apply. Although Eq. (1) was proved to be correct by De Josselin de Jong (1976) using the laws of friction, the validity of Eq. (2) was never checked. This must be done before the application Eq. (2) to cohesive-frictional materials. In this paper, we will show that Eq. (2) is not correct and then propose a correct stress-dilatancy relation, derived using the laws of friction used by De Josselin de Jong in his 1976 Géotechnique paper.

2 LAWS OF FRICTION

For a cohesive-frictional material, the shaer strength consists of frictional and cohesional componenta. The frictional component develops at the contact areas between the particles. Conceptually, it can be analyzed using the analogy of rigid blocks in contact, for which the interface friction angle is ϕ_c. If the resultant force on the interface of the two blocks makes an angle λ with the normal to the contact surface, then we can formulate (as De Joseelin de Jong (1976) did) the following laws of friction:

$$\lambda \begin{cases} < \phi_c \Rightarrow \text{there is no sliding} \\ = \phi_c \Rightarrow \text{sliding is either imminent or underway} \\ > \phi_c : \text{not possible} \end{cases} \tag{3}$$

When using the laws of friction for cohesive-frictional materials, it is necessary to transform the normal stresses following Caquot (1934):

$$\sigma^* = \sigma + c\cot\phi \tag{4}$$

Note that the shear stresses are not affected by this transformation (i.e., $\tau^* = \tau$). As shown in Figure 1, Eq. (4) maps the normal stresses σ into new, transformed normal stresses σ^* by shifting the shear stress axis (and thus the abscissa of the origin of the normal stress axis) so that the Mohr-Coulomb yield envelope passes through the origin of the new system, thereby eliminating the cohesive intercept from the equation for the envelope in $\sigma^* - \tau^*$ space.

3 ROWE'S SAW-TOOTH MODEL

Let us consider a cylindrical sample of cemented sand with height h and cross-sectional area A shown in Figure 2. In a triaxial compression test, the stresses acting on the sample boundaries are the principal stresses $\sigma_1 > \sigma_2 = \sigma_3$ shown in Figure 2(a). In Figure 2(b), an equivalent representation of the sample and its loading is shown. In it, the transformed stresses $\sigma^*_1 = \sigma_1 + c\cot\phi$ and $\sigma^*_3 = \sigma_3 + c\cot\phi$ are shown applied to a sample identical to the one in Figure 2(a) except for one detail: the sample in Figure 2(b) is uncemented. The cohesive intercept c is incorporated via the demonstrated stress transformation and there is no additional intercept. The meaning of Figure 2 is that the effects of the cohesive intercept c may be modeled considering an equivalent soil with the same friction angle ϕ but with $c = 0$ to which an isotropic stress $c\cot\phi$ is applied, in addition to any other loadings.

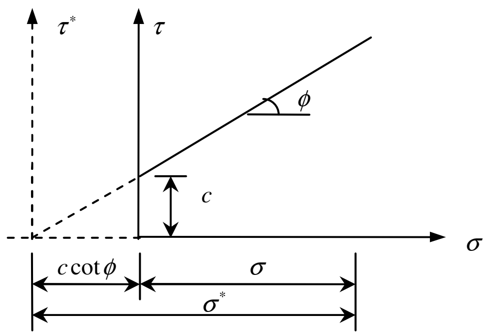

Figure 1. Visualization of stress transformation

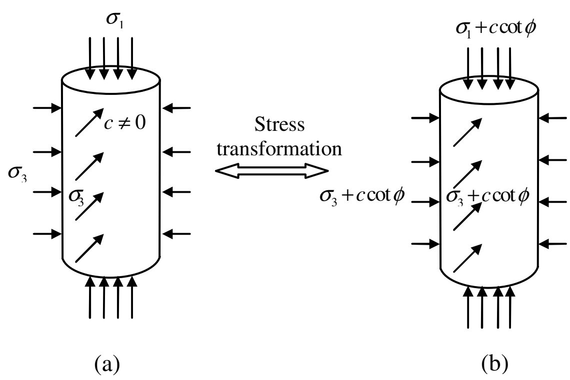

Figure 2. Stress transformation in axisymmetric condition

Using Rowe's saw-tooth model, the sliding occurs along separation planes between adjacent conglomerates of particles. As shown in Figure 3, the separation plane has a stepped, saw-tooth surface, and the direction of sliding is in the direction of teeth. After sliding, a gap opens between the teeth, leading to an increase in the volume of the sample. The separation plane makes an angle α with the minor principal stress σ_3, and the teeth make an angle β with the major principal stress σ_1. The angles α and β for all separation planes are assumed to be the same. The deviation angle between the teeth and the separation plane is

$$\theta = \alpha + \beta - \frac{\pi}{2} \tag{5}$$

Positive values of θ, as shown in Figure 3, indicate volume increase.

The force transmitted through the teeth is denoted by F and can be decomposed into vertical and horizontal components F_v and F_h, which are given by

$$F_v = (\sigma_1 + c\cot\phi) A \tag{6}$$

and

$$F_h = (\sigma_3 + c\cot\phi) A \tan\alpha \tag{7}$$

According to Figure 3, the normal 'n' line makes an angle β with the horizontal, and the force F deviates from the 'n' line by an angle λ. So F makes an angle $\beta + \lambda$ with the horizontal. The tangent of this angle follows directly from Eqs. (6) and (7):

$$\tan(\beta + \lambda) = \frac{F_v}{F_h} = \frac{\sigma_1 + c\cot\phi}{(\sigma_3 + c\cot\phi)\tan\alpha} \tag{8}$$

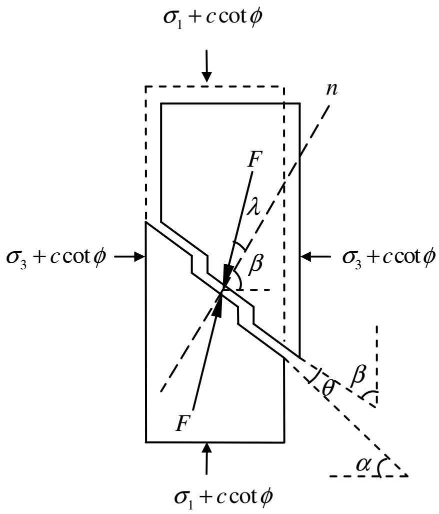

Figure 3. Forces on teeth in a separation plane

The Mohr-Coulomb criterion can be expressed as

$$\sigma_1 = \sigma_3 \tan^2\left(\frac{\pi}{4} + \frac{\phi}{2}\right) + 2c\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right) \tag{9}$$

or, in a more useful form, as

$$\frac{\sigma_1 + c\cot\phi}{\sigma_3 + c\cot\phi} = \frac{1 + \sin\phi}{1 - \sin\phi} = N \tag{10}$$

where N is referred to as the flow number. Using the flow number N, Eq. (8) can then be rewritten as

$$N = \frac{\sigma_1 + c\cot\phi}{\sigma_3 + c\cot\phi} = \tan\alpha\tan(\beta + \lambda) \tag{11}$$

The strain increment ratio D can also be expressed in terms of α and β. De Josselin de Jong (1976) showed that:

$$D = \frac{dV_h}{-dV_v} = 1 - \frac{d\varepsilon_v}{d\varepsilon_1} = \tan\alpha\tan\beta \tag{12}$$

where dV_h and dV_v are the volume change increments due to the horizontal and vertical displacement, respectively. The negative sign in front of dV_v is in keeping with the geomechanics convention of having contraction and shortening be positive. D is often called the dilatancy rate, and is connected to the dilation angle ψ through the following expression,

$$D = \frac{1 + \sin\psi}{1 - \sin\psi} \tag{13}$$

The assumption was made in the derivation of (12) that the total volume change increment dV consists of the algebraic sum of dV_h and dV_v. Because dV and dV_v are relatively easily obtained from measurements in triaxial tests, dV_h can be obtained by subtracting dV_v from the total volume change increment dV for either small or large strains. Thus, Eq. (12) applies to both small and large strains. As De Josselin de Jong (1976) and Rowe (1962) pointed, Eq. (12) is applicable to loadings in which the principal stresses and principal strain increments are coaxial.

4 FLOW RULE

For the saw-tooth model discussed in the previous section, the magnitudes of both N and D can be measured in triaxial compression tests. If D and N are known, Eqs. (11) and (12) contain three unknowns: α, β and λ. To solve for the unknowns, we need a third equation. The friction laws introduced as Eq. (3) provide us with this third equation: $\lambda_{max} = \phi_c$. This equation states that, for sliding to occur, the inclination λ of the resultant with respect to the normal to the plane of the teeth must match the critical-state friction angle ϕ_c. Before we use this equation, the angle α can be eliminated from (11) and (12) by introducing E^*, which is the quotient of N and D,

$$E^* = \frac{N}{D} = \frac{\tan(\beta + \lambda)}{\tan\beta} \tag{14}$$

E^* differs from Rowe's energy rate $\dot{E}$, which was defined as the ratio of work in to work out of the system. It may be obtained from test observations. Solving for λ:

$$\lambda = \tan^{-1}(E^* \tan\beta) - \beta \tag{15}$$

We must now find the maximum value of λ so we can make it equal to ϕ_c to satisfy the friction law. To find the value of β that maximizes λ_{max}, which we will denote as β_m, we first differentiate both sides of Eq. (15) with respect to β, obtaining:

$$\frac{\partial\lambda}{\partial\beta} = \frac{1}{1 + (E^* \tan\beta)^2} \cdot \frac{E^*}{\cos^2\beta} - 1 \tag{16}$$

We then substitute Eq. (14) into (16) and make $\partial\lambda / \partial\beta = 0$:

$$\beta_m = \frac{\pi}{4} - \frac{\lambda_{max}}{2} = \frac{\pi}{4} - \frac{\phi_c}{2} \tag{17}$$

Using the values of λ_{max} and β_m in Eq. (14) yields:

$$E^* = \frac{N}{D} = \tan^2\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right) = \frac{1 + \sin\phi_c}{1 - \sin\phi_c} \tag{18}$$

Eq. (18) can be further simplified to

$$N = DN_c \tag{19}$$

by using the flow number N_c at critical state, defined as

$$N_c = \frac{1 + \sin\phi_c}{1 - \sin\phi_c} \tag{20}$$

Eq. (19) is the correct form of the stress-dilatancy relation for cohesive-frictional materials. An example of a study presented in the literature that correctly accounted for the effects of cohesion c is that of Lade and Overton (1989), who studied the stress-strain behaviour of cemented sands using Eq. (1) (and not Eq. (2)). They relied on Caquot's stress transformation principle to account for the effects of cohesion, so their work is consistent with the principles presented in this paper.

Using Rowe's notation for the principal stress ratio, $R = \sigma_1/\sigma_3$, and rearranging Eq. (10):

$$R = N + \frac{2c}{\sigma_3}\sqrt{N} \quad (21)$$

Substituting Eq. (19) into (21):

$$R = DN_c + \frac{2c}{\sigma_3}\sqrt{DN_c} \quad (22)$$

Let us now compare Eq. (2) and (22). For that, Eq. (2) can be rewritten in terms of R, D and N_c:

$$R = DN_c + \frac{2c}{\sigma_3}D\sqrt{N_c} \quad (23)$$

By comparing Eq. (22) and (23), we see that the second term on the right side of Eq. (23) is too large by a factor $\sqrt{D}$.

The correct form of the stress-dilatancy relation to use in the Rowe framework is thus Eq. (22), which can also be expressed in terms of other stress and strain rate variables. For example:

$$\frac{\sigma_1}{\sigma_3} = \left(1 - \frac{d\varepsilon_v}{d\varepsilon_1}\right)\tan^2\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right) + \frac{2c}{\sigma_3}\tan\left(\frac{\pi}{4} + \frac{\phi_c}{2}\right)\sqrt{1 - \frac{d\varepsilon_v}{d\varepsilon_1}} \quad (24)$$

On examination of Eq. (24), we can see that without the incremental volume change, that is, with $d\varepsilon_v = 0$, Eq. (24) is simply the Mohr-Coulomb criterion with $\phi = \phi_c$. So the stress-dilatancy relation may be seen as a generalization of the Mohr-Coulomb criterion to include the effect of dilatancy rates on the friction angle ϕ.

In axisymmetric conditions, stresses are often expressed in terms of the mean p and deviatoric q stress variables, defined as:

$$p = (\sigma_1 + 2\sigma_3)/3 \quad (25)$$

and

$$q = \sigma_1 - \sigma_3 \quad (26)$$

The stress ratio η is then written as:

$$\eta = \frac{q}{p} \quad (27)$$

and the dilatancy rate as:

$$d = \frac{d\varepsilon_v^p}{d\varepsilon_s^p} \quad (28)$$

where $d\varepsilon_v^p$ and $d\varepsilon_s^p$ are the plastic volumetric strain rate and plastic deviatoric strain rate, respectively. In terms of these variables, Eq. (22) is rewritten as:

$$d = \frac{9(M - \eta) - 3m_c}{9 + M(3 - 2\eta) + m_c} \quad (29)$$

where M is the material constant related to the critical-state friction angle ϕ_c and m_c is a term related to the cohesive intercept c, which is given by:

$$m_c = \frac{6(3 - M)(c/p)^2}{3 - \eta} - \frac{2c(3 - M)}{p}\sqrt{\left(\frac{3c/p}{3 - \eta}\right)^2 + \frac{3 + 2\eta}{3 - \eta}} \quad (30)$$

5 CONCLUSIONS

In this paper, we derived a correct stress-dilatancy relationship for cemented sands. We did so in the transformed stress space, in which the material has the same ϕ but no cohesion or tensile strength. Application of a uniform hydrostatic stress field $c\cot\phi$ to the material compensates for making $c = 0$. The derivation further relies on use of Rowe's saw-tooth model together with the application of the laws of friction. Due to the incorrect hypothesis originally made, Eq. (2) (or its equivalent form, Eq. (23)) is incorrect. The correct equation to use in modeling cemented sands is Eq. (22) or its equivalent forms expressed in terms of other stress and strain variables, Eqs. (24) and (29).

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