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Output Feedback Control Synthesis and Stabilization for Positive Polynomial Fuzzy Systems Under L_1 Performance

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Abstract. This paper presents the stabilization for positive nonlinear systems using polynomial fuzzy models. To conform better to the practical scenarios that system states are not completely measurable, the static output feedback (SOF) control strategy instead of the state feedback control method is employed to realize the stability and positivity of the positive polynomial fuzzy system (PPFS) with satisfying L_1-induced performance. However, some troublesome problems in analysis and control design will follow, such as the non-convex problem. Fortunately, by doing mathematical tricks, the non-convex problem is skillfully dealt with. Furthermore, the neglect of external disturbances may lead to a great negative impact on the performance of positive systems. For the sake of guaranteeing the asymptotic stability and positivity under the satisfaction of the optimal performance of the PPFS, it is significant to take the L_1-induced performance requirement into consideration as well. In addition, a linear co-positive Lyapunov function is chosen so that the positivity can be extracted well and the stability analysis becomes simple. By using the sum of squares (SOS) technique, the convex stability and positivity conditions in the form of SOS are derived. Eventually, for illustrating the advantages of the proposed method, a simulation example is shown in the simulation section.

Keywords. positive polynomial fuzzy system (PPFS), static output feedback (SOF), L_1 performance, stability analysis, sum of squares (SOS)

1. Introduction

As a particular kind of systems, positive systems have attracted ever-increasing researchers to conduct a deep-going research on the stability and positivity analysis. In fact, a great number of practical systems in various disciplines, for instance, physiology [1], communication networks [2] and biology [3] belong to positive systems because the system states maintain in the positive quadrant with the non-negative initial conditions. However, due to the special property of positive systems, a good deal of challenging and

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interesting problems cannot be coped with by employing some mature techniques for general systems [4]. From this point, a series of elegant properties of positive systems are worth studying and digging [5]. Hence, a growing number of researchers have shown great interest in this research topic.

Up to now, many valuable research results have been obtained, which lay a good foundation for future research on positive systems. In [6], the stability analysis for positive linear systems with time-varying delays was carried out. In [7], the authors designed positive filters for positive systems to reduce the influence of the external disturbance. It can be seen that most of existing results are for positive linear systems because the system structures are simple, meanwhile, the controller design is easy to realize. Nevertheless, it is generally acknowledged that a lot of actual systems demonstrate nonlinear characteristics rather than linear ones in practical applications. Because of the complexity of positive nonlinear systems, some current results for positive linear systems cannot be directly employed for positive nonlinear systems. Therefore, it is worth a try to study the control synthesis for positive nonlinear systems.

Fuzzy-model-based control theory offers a systematic way to deal with analysis and control synthesis for nonlinear systems. One of the well-known approaches is through Takage-Sugeno (T-S) fuzzy model which is in the light of a set of fuzzy rules to express a global nonlinear system [8]. With the further study of fuzzy theory, polynomial fuzzy models have been put forward, comparing with T-S fuzzy models, this kind of fuzzy models have many distinct merits. First of all, more complex and extensive nonlinear systems are able to be handled by polynomial fuzzy models since not only constant terms but also polynomials are permitted in the membership functions (MFs) as well as the system matrices [9]. On the other hand, the imperfect premise matching (IPM) concept and membership-function-dependent (MFD) analysis techniques have been proposed for polynomial fuzzy-model-based control theory [10], which have obvious advantages than parallel distributed compensation (PDC) scheme and membership-function-independent (MFI) analysis technique. For instance, the fuzzy controllers can be designed flexibly and implemented simplier. Besides, MFD analysis techniques are of great help to reduce the conservativeness of the results. For all these reasons, using a polynomial fuzzy model to handle complex nonlinear systems is a better way to facilitate the control synthesis and stability analysis. However, positive polynomial fuzzy systems (PPFSs) have many differences from the general systems, which means some significant results for general polynomial fuzzy systems are difficult to be used for PPFSs [11]. Thereby, the controller design and stabilization for PPFSs is a meaningful but challenging research topic.

In resent literature, a few results corresponding to control synthesis for the positive systems have been provided. It is worth noting that these results are on the basis of state feedback control strategy instead of static output feedback (SOF) control method. Nevertheless, from the practical point of view, it makes more sense to design the SOF controllers for PPFSs because this kind of controllers do not require full state information of PPFSs, meanwhile, it is simple and money-saving to put into practice. In this regard, designing SOF fuzzy controllers for PPFSs is more realistic and reasonable. Unfortunately, some thorny problems will follow, for instance, non-convex terms usually are able to exist in stability and positivity conditions [12]. Although some marture techniques have been provided [13,14], these methods are just appropriate for genneral systems rather than positive polynomial fuzzy systems. Due to this barrier, the obtained results based on SOF control approach for PPFSs are relatively few. Considering the solution of non-convex problem for positive polynomial fuzzy systems, we have proposed a method to transform non-convex terms into convex ones in [12], but the L_1-induced performance was not taken into consideration. It is well known that in engineering applications, some practical systems are required to meet the performance requirements. Generally speaking, L_1 performance can better capture the positivity of PPFSs since L_1-norm represents the sum of the values of the components. Therefore, to accord with the practical scenarios, L_1 performance is considered as well so that the closed-loop PPFSs can satisfy the stability and positivity under L_1-induced performance requirement. Due to the introduction of L_1 performance, the techniques given in [12] are not suitable for the new non-convex problem in this paper. Looking for an appropriate approach to solve this issue also is an inspiration for us to carry out this work.

2. Preliminaries

2.1. Notation

The monomial in $\mathbf{x}(t) = [x_1(t),\ldots,x_n(t)]^T$ is defined as $x_1^{d_1}(t),\ldots,x_n^{d_n}(t)$, where d_k, $k \in \{1,\ldots,n\}$, is a non-negative integer. The degree of a monomial is defined as $d = \sum_{k=1}^{n} d_k$. A polynomial $\mathbf{p}(\mathbf{x}(t))$ is shown as finite linear combination of monomials with real coefficients. If a polynomial $\mathbf{p}(\mathbf{x}(t))$ is able to be represented as $\mathbf{p}(\mathbf{x}(t)) = \sum_{j=1}^{m} \mathbf{q}_j(\mathbf{x}(t))^2$, where m is a non-zero positive integer and $\mathbf{q}_j(\mathbf{x}(t))$ is a polynomial for all j, we can draw a conclusion that $\mathbf{p}(\mathbf{x}(t)) \geq 0$ is a SOS. For a matrix $\mathbf{N} \in \Re^{m\times n}$, where n_{rs} denotes the element located at the r-th row and s-th column. $\mathbf{N} \succeq 0$, $\mathbf{N} \succ 0$, $\mathbf{N} \preceq 0$ and $\mathbf{N} \prec 0$ mean that each element n_{rs} is non-negative, positive, non-positive and negative, respectively. $\mathbf{Q}(\mathbf{x}) = \mathrm{diag}(x_1,\ldots,x_n)$ means that $\mathbf{Q}(\mathbf{x})$ is a diagonal matrix with all of the diagonal elements being $x_1,\ldots,x_n$.

2.2. Positive Polynomial Fuzzy Model

A p-rule positive polynomial fuzzy model is shown:

$$\text{Rule } i: \text{IF } f_1(\mathbf{x}(t)) \text{ is } M_1^i \text{ AND} \cdots \text{AND } f_\Psi(\mathbf{x}(t)) \text{ is } M_\Psi^i$$

$$\text{THEN} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t), \\ \mathbf{z}(t) = \mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{D}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{D}_{i\omega}\tilde{\mathbf{w}}(t), \\ \mathbf{y}(t) = \mathbf{E}\mathbf{x}(t) + \mathbf{E}_\omega\tilde{\mathbf{w}}(t), \end{cases} \tag{1}$$

where $\mathbf{u}(t) \in \Re^m$, $\tilde{\mathbf{w}}(t) \in \Re^h$, $\mathbf{z}(t) \in \Re^q$ and $\mathbf{y}(t) \in \Re^l$ are the system state vector, the input vector, the disturbance signal, the measurement output and the controlled output, respectively; $\mathbf{A}_i(\mathbf{x}(t))$, $\mathbf{B}_i(\mathbf{x}(t))$, $\mathbf{B}_{i\omega}$, $\mathbf{C}_i(\mathbf{x}(t))$, $\mathbf{D}_i(\mathbf{x}(t))$, $\mathbf{D}_{i\omega}$, $\mathbf{E}$ and $\mathbf{E}_\omega$ are the system matrices with appropriate dimensions.

The overall dynamics of the PPFS is introduced:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{A}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{B}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{B}_{i\omega}\tilde{\mathbf{w}}(t)\big), \\ \mathbf{z}(t) = \sum_{i=1}^{p} w_i(\mathbf{x}(t))\big(\mathbf{C}_i(\mathbf{x}(t))\mathbf{x}(t) + \mathbf{D}_i(\mathbf{x}(t))\mathbf{u}(t) + \mathbf{D}_{i\omega}\tilde{\mathbf{w}}(t)\big), \\ \mathbf{y}(t) = \mathbf{E}\mathbf{x}(t) + \mathbf{E}_\omega\tilde{\mathbf{w}}(t), \end{cases} \tag{2}$$

where $w_i(\mathbf{x}(t))$ is the normalized grade of membership with satisfying $\sum_{i=1}^{p} w_i(\mathbf{x}(t)) = 1, w_i(\mathbf{x}(t)) \geq 0\ \forall i$.

In order to have a better understanding of positive systems, we give some definitions and lemmas before designing the SOF polynomial fuzzy controller.

Definition 1 *[15] A system is called a positive system if the corresponding trajectory* $\mathbf{x}(t) \succeq 0$ *for all* $t \geq 0$ *is held under the initial condition* $\mathbf{x}(0) = \mathbf{x}_0 \succeq 0$.

Definition 2 *[15] If the off-diagonal elements in matrix* $\mathbf{M}$ *are non-negative:* $m_{rs} \succeq 0$, $r \neq s$, *then this matrix is a Metzler matrix.*

Lemma 1 *[16] System* (2) *is a positive system if* $\mathbf{A}_i(\mathbf{x}(t))$ *is a Metzler matrix,* $\mathbf{B}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{B}_{i\omega} \succeq 0$, $\mathbf{C}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{D}_i(\mathbf{x}(t)) \succeq 0$, $\mathbf{D}_{i\omega} \succeq 0$, $\mathbf{E} \succeq 0$ *and* $\mathbf{E}_\omega \succeq 0$.

2.3. Polynomial Fuzzy Controller Design

By using IPM technique, a c-rule SOF polynomial fuzzzy controller is designed:

$$\text{Rule } j: \text{IF } g_1(\mathbf{y}(t)) \text{ is } N_1^j \text{ AND} \ldots \text{AND } g_\Omega(\mathbf{y}(t)) \text{ is } N_\Omega^j$$
$$\text{THEN } \mathbf{u}(t) = \mathbf{K}_j \mathbf{y}(t), \tag{3}$$

where $\mathbf{K}_j \in \Re^{m \times l}$ is the SOF gain to be determined.

Through recalling the expression of $\mathbf{y}(t)$, we have:

$$\mathbf{u}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\mathbf{K}_j\mathbf{y}(t) = \sum_{j=1}^{c} m_j(\mathbf{y}(t))\big(\mathbf{K}_j\mathbf{E}\mathbf{x}(t) + \mathbf{K}_j\mathbf{E}_\omega\tilde{\mathbf{w}}(t)\big), \tag{4}$$

where $m_j(\mathbf{y}(t))$ is the normalized grade of membership with satisfying $\sum_{j=1}^{c} m_j(\mathbf{y}(t)) = 1, m_j(\mathbf{y}(t)) \geq 0, \forall j$.

Remark 1 *It is worth noting that the IPM method is employed to design the SOF controller because it can make the controller design more flexible and the implementation cost more economical.*

To simplify, t is omitted in the rest parts, which means $\mathbf{x}(t)$ and $\mathbf{y}(t)$ will be abbreviated as $\mathbf{x}$ and $\mathbf{y}$, respectively.

3. Stability and Positivity Analysis under L_1 Performance

In this section, we keep our mind on analyzing the stability and positivity for closed-loop PPFSs with satisfying L_1 performance index. A linear co-positive Lyapunov function is chosen to promote the stability and positivity analysis. Meanwhile, convex SOS-based conditions are derived by employing some useful techniques to solve non-convex terms.

3.1. SOF Positive Polynomial Fuzzy Control Systems

In terms of the PPFS (2) and the SOF polynomial fuzzy controller (4), the SOF positive polynomial fuzzy control system is obtained:

$$\begin{cases} \dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\big(\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}\big)\mathbf{x}+\big(\mathbf{B}_{i\omega}+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}_\omega\big)\tilde{\mathbf{w}}\Big) \\ = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\tilde{\mathbf{A}}_{ij}(\mathbf{x})\mathbf{x}+\tilde{\mathbf{B}}_{ij}(\mathbf{x})\tilde{\mathbf{w}}\Big), \\ \mathbf{z} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\big(\mathbf{C}_i(\mathbf{x})+\mathbf{D}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}\big)\mathbf{x}+\big(\mathbf{D}_{i\omega}+\mathbf{D}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}_\omega\big)\tilde{\mathbf{w}}\Big) \\ = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\Big(\tilde{\mathbf{C}}_{ij}(\mathbf{x})\mathbf{x}+\tilde{\mathbf{D}}_{ij}(\mathbf{x})\tilde{\mathbf{w}}\Big), \\ \mathbf{y} = \mathbf{E}\mathbf{x}+\mathbf{E}_\omega\tilde{\mathbf{w}}. \end{cases} \tag{5}$$

Remark 2 *According to Lemma 1, the SOF positive polynomial fuzzy control system* (5) *is a positive system if* $\tilde{\mathbf{A}}_{ij}(\mathbf{x})$ *is a Metzler matrix,* $\tilde{\mathbf{B}}_{ij}(\mathbf{x}) \succeq 0$, $\tilde{\mathbf{C}}_{ij}(\mathbf{x}) \succeq 0$, $\tilde{\mathbf{D}}_{ij}(\mathbf{x}) \succeq 0$, $\mathbf{E} \succeq 0$, $\mathbf{E}_\omega \succeq 0$, *for all i, j.*

Next, the L_1-induced performance is introduced to facilitate the analysis process.

Definition 3 *[17] The system* (5) *can satisfy* L_1*-induced performance at the level* γ*, if the following inequality can be ensured with satisfying zero initial conditions*

$$||\mathbf{z}||_{L_1} < \gamma||\tilde{\mathbf{w}}||_{L_1}, \tag{6}$$

where γ *is the optimal level to be determined.*

3.2. Stability Analysis of SOF Positive Polynomial Fuzzy Control Systems

In order to better capture the positivity of the SOF positive polynomial fuzzy control system (5), a linear co-positive Lyapunov function candidate [16] is employed to establish some stability and positivity criteria:

$$V(t) = \lambda^T\mathbf{x}, \tag{7}$$

where $\lambda = [\lambda_1,\ldots,\lambda_n]^T \succ 0$ is a vector to be determined.

The $\dot{V}(t)$ is given as follows:

$$\dot{V}(t) = \lambda^T\dot{\mathbf{x}} = \sum_{i=1}^{p}\sum_{j=1}^{c} w_i(\mathbf{x})m_j(\mathbf{y})\lambda^T\Big(\tilde{\mathbf{A}}_{ij}(\mathbf{x})\mathbf{x}+\tilde{\mathbf{B}}_{ij}(\mathbf{x})\tilde{\mathbf{w}}\Big). \tag{8}$$

In the following, by recalling the definition (6), the L_1 performance index is shown:

$$\begin{aligned} J &= \int_0^\infty ||\mathbf{z}||_{L_1} - \gamma||\tilde{\mathbf{w}}||_{L_1} d_t = \int_0^\infty ||\mathbf{z}||_{L_1} - \gamma||\tilde{\mathbf{w}}||_{L_1} + \dot{V} - \dot{V} d_t \\ &= \int_0^\infty \sum_{k=1}^{q}\mathbf{z} - \gamma\sum_{k=1}^{p}\tilde{\mathbf{w}} + \dot{V} d_t - V(\infty) + V(0) \\ &= \int_0^\infty \mathbf{I}_1^T\mathbf{z} - \gamma\mathbf{I}_2^T\tilde{\mathbf{w}} + \dot{V} d_t - V(\infty) + V(0), \end{aligned} \tag{9}$$

where $V(\infty)$ is equal to 0 when $t \to \infty$, meanwhile, the unitial condition $V(0)$ is zero. $\mathbf{I}_1 \in \Re^q$ and $\mathbf{I}_2 \in \Re^p$ are vectors with all of the elements being 1.

Therefore, taking the expressions of $\mathbf{z}$ and $\dot{V}$ into (9), we have:

$$
\begin{aligned}
J &= \int_0^\infty \mathbf{I}_1^T \mathbf{z} - \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \dot{V} d_t \\
&= \int_0^\infty \mathbf{I}_1^T \Big(\sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \big(\tilde{\mathbf{C}}_{ij}(\mathbf{x})\mathbf{x} + \tilde{\mathbf{D}}_{ij}(\mathbf{x})\tilde{\mathbf{w}} \big) \Big) \\
&- \gamma \mathbf{I}_2^T \tilde{\mathbf{w}} + \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \lambda^T \big(\tilde{\mathbf{A}}_{ij}(\mathbf{x})\mathbf{x} + \tilde{\mathbf{B}}_{ij}(\mathbf{x})\tilde{\mathbf{w}} \big) d_t \\
&= \int_0^\infty \sum_{i=1}^{p} \sum_{j=1}^{c} w_i(\mathbf{x}) m_j(\mathbf{y}) \Big(\big(\mathbf{I}_1^T \tilde{\mathbf{D}}_{ij}(\mathbf{x}) - \gamma \mathbf{I}_2^T + \lambda^T \tilde{\mathbf{B}}_{ij}(\mathbf{x}) \big) \tilde{\mathbf{w}} \\
&+ \big(\mathbf{I}_1^T \tilde{\mathbf{C}}_{ij}(\mathbf{x}) + \lambda^T \tilde{\mathbf{A}}_{ij}(\mathbf{x}) \big) \mathbf{x} \Big) d_t. \qquad (10)
\end{aligned}
$$

In order to make it easier to explain , we define:

$$
\begin{aligned}
\mathbf{Q}_{1ij}(\mathbf{x}) &= \mathbf{I}_1^T \tilde{\mathbf{D}}_{ij}(\mathbf{x}) - \gamma \mathbf{I}_2^T + \lambda^T \tilde{\mathbf{B}}_{ij}(\mathbf{x}) \\
&= \mathbf{I}_1^T \big(\mathbf{D}_{i\omega} + \mathbf{D}_i(\mathbf{x})\mathbf{K}_j \mathbf{E}_\omega \big) - \gamma \mathbf{I}_2^T + \lambda^T \big(\mathbf{B}_{i\omega} + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j \mathbf{E}_\omega \big) \\
&= \mathbf{I}_1^T \mathbf{D}_{i\omega} + \lambda^T \mathbf{B}_{i\omega} - \gamma \mathbf{I}_2^T + \big(\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \lambda^T \mathbf{B}_i(\mathbf{x}) \big) \mathbf{K}_j \mathbf{E}_\omega, \qquad (11) \\
\mathbf{Q}_{2ij}(\mathbf{x}) &= \mathbf{I}_1^T \tilde{\mathbf{C}}_{ij}(\mathbf{x}) + \lambda^T \tilde{\mathbf{A}}_{ij}(\mathbf{x}) \\
&= \mathbf{I}_1^T \big(\mathbf{C}_i(\mathbf{x}) + \mathbf{D}_i(\mathbf{x})\mathbf{K}_j \mathbf{E} \big) + \lambda^T \big(\mathbf{A}_i(\mathbf{x}) + \mathbf{B}_i(\mathbf{x})\mathbf{K}_j \mathbf{E} \big) \\
&= \mathbf{I}_1^T \mathbf{C}_i(\mathbf{x}) + \lambda^T \mathbf{A}_i(\mathbf{x}) + \big(\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \lambda^T \mathbf{B}_i(\mathbf{x}) \big) \mathbf{K}_j \mathbf{E}. \qquad (12)
\end{aligned}
$$

From (10), it can be seen that $J < 0$ can be guaranteed by $\mathbf{Q}_{1ij}(\mathbf{x}) \prec 0$ and $\mathbf{Q}_{2ij}(\mathbf{x}) \prec 0$ for all i and j. Regrettably, there are non-convex terms $\lambda^T \mathbf{B}_i(\mathbf{x})\mathbf{K}_j \mathbf{E}_\omega$ and $\lambda^T \mathbf{B}_i(\mathbf{x})\mathbf{K}_j \mathbf{E}$ in (11) and (12), respectively. Hence, our attention should be focused on transforming the non-convex terms into convex ones in the following.

In accordance with (11) and (12), we find if $\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \lambda^T \mathbf{B}_i(\mathbf{x}) \succeq \mathbf{I}_m^T$ and $\mathbf{K}_j \prec 0$ are satisfied, the non-convex terms can be dealt with as:

$$
\big(\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \lambda^T \mathbf{B}_i(\mathbf{x}) \big) \mathbf{K}_j \mathbf{E}_\omega \preceq \mathbf{I}_m^T \mathbf{K}_j \mathbf{E}_\omega, \qquad (13)
$$

$$
\big(\mathbf{I}_1^T \mathbf{D}_i(\mathbf{x}) + \lambda^T \mathbf{B}_i(\mathbf{x}) \big) \mathbf{K}_j \mathbf{E} \preceq \mathbf{I}_m^T \mathbf{K}_j \mathbf{E}. \qquad (14)
$$

where $\mathbf{I}_m^T \in \Re^m$ is a column vector with all the elements being 1:

Now, by introducing (13) and (14) into (11) and (12), respectively, the convex stability conditions are derived:

$$
\mathbf{Q}_{1ij}(\mathbf{x}) \preceq \mathbf{I}_1^T \mathbf{D}_{i\omega} + \lambda^T \mathbf{B}_{i\omega} - \gamma \mathbf{I}_2^T + \mathbf{I}_m^T \mathbf{K}_j \mathbf{E}_\omega \prec 0, \qquad (15)
$$

$$
\mathbf{Q}_{2ij}(\mathbf{x}) \preceq \mathbf{I}_1^T \mathbf{C}_i(\mathbf{x}) + \lambda^T \mathbf{A}_i(\mathbf{x}) + \mathbf{I}_m^T \mathbf{K}_j \mathbf{E} \prec 0. \qquad (16)
$$

After the above analysis, the non-convex problem has been addressed well. The next task is to carry out the positivity analysis. By recalling Remark 2, the positivity conditions should be as follows:

$$\mathbf{A}_i(\mathbf{x})+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j\mathbf{E} \quad \text{is a Metzler, } \mathbf{B}_{i\omega}+\mathbf{B}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}_\omega \succeq 0,$$
$$\mathbf{C}_i(\mathbf{x})+\mathbf{D}_i(\mathbf{x})\mathbf{K}_j\mathbf{E} \succeq 0,\ \mathbf{D}_{i\omega}+\mathbf{D}_i(\mathbf{x})\mathbf{K}_j\mathbf{E}_\omega \succeq 0. \tag{17}$$

In light of (17), it is worth noting that the positivity conditions are convex, therefore, all convex stability and positivity conditions have been established. In the following, the analyzed results are summarized in Theorem 1.

Theorem 1 *Given that the positive polynomial fuzzy model* (2) *with satisfying Lemma 1 can guarantee the stability and positivity under L_1-induced performance by the SOF polynomial fuzzy controller* (5) *if there exist gain matrix $\mathbf{K}_j \in \Re^{m\times l}$, vector $\lambda \in \Re^n$ and the optimal performance index γ such that the SOS-based positivity and stability conditions are satisfied:*

$$\upsilon^T\left(a_{irs}(\mathbf{x})+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_s\right)\upsilon \quad \text{is } SOS\ \forall\, i,j,r\neq s; \tag{18}$$
$$\upsilon^T\left(b_{i\omega rs}+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_{\omega s}\right)\upsilon \text{ is } SOS\ \forall\, i,j,r,s; \tag{19}$$
$$\upsilon^T\left(c_{irs}(\mathbf{x})+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_s\right)\upsilon \text{ is } SOS\ \forall\, i,j,r,s; \tag{20}$$
$$\upsilon^T\left(d_{i\omega rs}+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_{\omega s}\right)\upsilon \text{ is } SOS\ \forall\, i,j,r,s; \tag{21}$$
$$\rho^T\left(diag\left(\lambda-\varepsilon_1\mathbf{I}_n\right)\right)\rho \text{ is } SOS; \tag{22}$$
$$-\sigma^T\left(diag\left(\mathbf{I}_1^T\mathbf{D}_{i\omega}+\lambda^T\mathbf{B}_{i\omega}-\gamma\mathbf{I}_2^T+\mathbf{I}_m^T\mathbf{K}_j\mathbf{E}_\omega+\varepsilon_2(\mathbf{x})\mathbf{I}_h^T\right)\right)\sigma \text{ is } SOS\ \forall\, i,j; \tag{23}$$
$$-\rho^T\left(diag\left(\mathbf{I}_1^T\mathbf{C}_i(\mathbf{x})+\lambda^T\mathbf{A}_i(\mathbf{x})+\mathbf{I}_m^T\mathbf{K}_j\mathbf{E}+\varepsilon_3(\mathbf{x})\mathbf{I}_n^T\right)\right)\rho \text{ is } SOS\ \forall\, i,j; \tag{24}$$
$$-\upsilon^T\left(k_{jrs}+\varepsilon_4\right)\upsilon \text{ is } SOS\ \forall\, j,r,s; \tag{25}$$
$$\mu^T\left(\mathbf{I}_1^T\mathbf{D}_i(\mathbf{x})+\lambda^T\mathbf{B}_i(\mathbf{x})-\mathbf{I}_m^T\right)\mu \text{ is } SOS\ \forall\, i. \tag{26}$$

where γ is the optimal index to be determined. υ is a arbitrary scalar and $\rho \in \Re^n$, $\sigma \in \Re^h$ and $\mu \in \Re^m$ are arbitrary vectors independent of $\mathbf{x}$ and $\mathbf{y}$; $\varepsilon_1 > 0$ and $\varepsilon_4 > 0$ are predefined scalars and $\varepsilon_2(\mathbf{x}) > 0$ and $\varepsilon_3(\mathbf{x}) > 0$ for $\mathbf{x} \neq 0$ are predefined scalar polynomials. $a_{irs}(\mathbf{x})$, $b_{i\omega rs}$, $c_{irs}(\mathbf{x})$ and $d_{i\omega rs}$ are the r-th row and s-th column element in $\mathbf{A}_i(\mathbf{x})$, $\mathbf{B}_{i\omega}$, $\mathbf{C}_i(\mathbf{x})$ and $\mathbf{D}_{i\omega}$, respectively. $\mathbf{b}_{ir}(\mathbf{x})$ and $\mathbf{d}_{ir}(\mathbf{x})$ are the r-th row vectors in $\mathbf{B}_i(\mathbf{x})$ and $\mathbf{D}_i(\mathbf{x})$, respectively. $\mathbf{e}_s$ and $\mathbf{e}_{\omega s}$ are the s-th column vectors in $\mathbf{E}$ and $\mathbf{E}_\omega$, respectively. k_{jrs} is the r-th row and s-th column element in $\mathbf{K}_j$ to be determined.

Remark 3 *From* (18) *to* (21)*, these conditions are able to ensure the positivity of the SOF positive polynomial fuzzy control systems. From* (22) *to* (26)*, these conditions can guarantee the stability with satisfying the L_1 performance index.*

Corollary 1 *Given that the positive polynomial fuzzy model* (2) *with satisfying Lemma 1 can guarantee the stability and positivity under L_1-induced performance by using PDC technique to design the SOF polynomial fuzzy controller* (5)*, if there exist gain matrix $\mathbf{K}_j \in \Re^{m\times l}$, vector $\lambda \in \Re^n$ and the optimal performance index γ such that the SOS-based positivity and stability conditions are satisfied:*

$$(22),(25)-(26)$$

$$\upsilon^T\left(a_{irs}(\mathbf{x})+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_i\mathbf{e}_s\right)\upsilon \quad \textit{is SOS}\ \forall\, i,r\neq s; \tag{27}$$

$$\upsilon^T\left(a_{irs}(\mathbf{x})+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_s+a_{jrs}(\mathbf{x})+\mathbf{b}_{jr}(\mathbf{x})\mathbf{K}_i\mathbf{e}_s\right)\upsilon \quad \textit{is SOS}\ \forall\, i<j,r\neq s; \tag{28}$$

$$\upsilon^T\left(b_{i\omega rs}+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_i\mathbf{e}_{\omega s}\right)\upsilon \ \textit{is SOS}\ \forall\, i,r,s; \tag{29}$$

$$\upsilon^T\left(b_{i\omega rs}+\mathbf{b}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_{\omega s}+b_{j\omega rs}+\mathbf{b}_{jr}(\mathbf{x})\mathbf{K}_i\mathbf{e}_{\omega s}\right)\upsilon \ \textit{is SOS}\ \forall\, i<j,r,s; \tag{30}$$

$$\upsilon^T\left(c_{irs}(\mathbf{x})+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_i\mathbf{e}_s\right)\upsilon \ \textit{is SOS}\ \forall\, i,r,s; \tag{31}$$

$$\upsilon^T\left(c_{irs}(\mathbf{x})+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_s+c_{jrs}(\mathbf{x})+\mathbf{d}_{jr}(\mathbf{x})\mathbf{K}_i\mathbf{e}_s\right)\upsilon \ \textit{is SOS}\ \forall\, i<j,r,s; \tag{32}$$

$$\upsilon^T\left(d_{i\omega rs}+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_i\mathbf{e}_{\omega s}\right)\upsilon \ \textit{is SOS}\ \forall\, i,r,s; \tag{33}$$

$$\upsilon^T\left(d_{i\omega rs}+\mathbf{d}_{ir}(\mathbf{x})\mathbf{K}_j\mathbf{e}_{\omega s}+d_{j\omega rs}+\mathbf{d}_{jr}(\mathbf{x})\mathbf{K}_i\mathbf{e}_{\omega s}\right)\upsilon \ \textit{is SOS}\ \forall\, i<j,r,s; \tag{34}$$

$$-\sigma^T\left(diag\left(\mathbf{I}_1^T\mathbf{D}_{i\omega}+\lambda^T\mathbf{B}_{i\omega}-\gamma\mathbf{I}_2^T+\mathbf{I}_m^T\mathbf{K}_i\mathbf{E}_\omega+\varepsilon_2(\mathbf{x})\mathbf{I}_h^T\right)\right)\sigma \ \textit{is SOS}\ \forall\, i; \tag{35}$$

$$-\sigma^T\Big(diag\big(\mathbf{I}_1^T\mathbf{D}_{i\omega}+\lambda^T\mathbf{B}_{i\omega}-\gamma\mathbf{I}_2^T+\mathbf{I}_m^T\mathbf{K}_j\mathbf{E}_\omega$$
$$+\mathbf{I}_1^T\mathbf{D}_{j\omega}+\lambda^T\mathbf{B}_{j\omega}-\gamma\mathbf{I}_2^T+\mathbf{I}_m^T\mathbf{K}_i\mathbf{E}_\omega+\varepsilon_2(\mathbf{x})\mathbf{I}_h^T\big)\Big)\sigma \ \textit{is SOS}\ \forall\, i<j \tag{36}$$

$$-\rho^T\left(diag\left(\mathbf{I}_1^T\mathbf{C}_i(\mathbf{x})+\lambda^T\mathbf{A}_i(\mathbf{x})+\mathbf{I}_m^T\mathbf{K}_i\mathbf{E}+\varepsilon_3(\mathbf{x})\mathbf{I}_n^T\right)\right)\rho \ \textit{is SOS}\ \forall\, i; \tag{37}$$

$$-\rho^T\Big(diag\big(\mathbf{I}_1^T\mathbf{C}_i(\mathbf{x})+\lambda^T\mathbf{A}_i(\mathbf{x})+\mathbf{I}_m^T\mathbf{K}_j\mathbf{E}$$
$$+\mathbf{I}_1^T\mathbf{C}_j(\mathbf{x})+\lambda^T\mathbf{A}_j(\mathbf{x})+\mathbf{I}_m^T\mathbf{K}_i\mathbf{E}+\varepsilon_3(\mathbf{x})\mathbf{I}_n^T\big)\Big)\rho \ \textit{is SOS}\ \forall\, i<j; \tag{38}$$

Remark 4 *In general, PDC technique is helpful to reduce the conservativeness, but there are some limitations of using this technique. For example, it greatly reduces the flexibility of controller design because PDC technique requires the polynomial fuzzy controller and the polynomial fuzzy system share same fuzzy rules, which means both the number and the type of the MFs should be same. Therefore, when the number and/or the type of the MFs of the polynomial fuzzy system are large and/or complex, it becomes hard to design and implement the polynomial fuzzy controller. Also, the MFI analysis is a source of conservativeness. See [9,10] for further details of IPM concept and MFD analysis.*

4. Simulation Example

4.1. Scenario

A positive polynomial fuzzy model with 3 fuzzy rules is presented:

$$\mathbf{A}_1(x_1)=\begin{bmatrix}0.03 & 0.45+0.08x_1^2\\ 0.98 & -1.13-x_1^2+0.13x_1\end{bmatrix},\mathbf{A}_2(x_1)=\begin{bmatrix}0.06 & 0.42+0.1x_1^2\\ 0.94 & -1.25-x_1^2+0.26x_1\end{bmatrix},$$

$$\mathbf{A}_3(x_1)=\begin{bmatrix}0.08 & 0.39+0.16x_1^2\\ 0.62 & -1.06-x_1^2+0.37x_1\end{bmatrix},\mathbf{B}_1=\begin{bmatrix}0.36\\ 0.22\end{bmatrix},\mathbf{B}_2=\begin{bmatrix}0.37\\ 0.24\end{bmatrix},\mathbf{B}_3=\begin{bmatrix}0.36\\ 0.18\end{bmatrix},$$

$$\mathbf{B}_{1w}=\begin{bmatrix}1.12\\ 1.18\end{bmatrix},\mathbf{B}_{2w}=\begin{bmatrix}1.13\\ 1.16\end{bmatrix},\mathbf{B}_{3w}=\begin{bmatrix}1.16\\ 1.14\end{bmatrix},\mathbf{x}=[x_1 \quad x_2]^T,\ \mathbf{E}=[1 \quad 0],\ \mathbf{E}_\omega=1,$$

$$\mathbf{C}_1(x_1)=\begin{bmatrix}1.07 & 1.15+0.23x_1^2\end{bmatrix},\mathbf{C}_2(x_1)=\begin{bmatrix}1.05 & 1.18+0.15x_1^2\end{bmatrix},$$

$$\mathbf{C}_3(x_1)=\begin{bmatrix}1.14 & 1.2+0.17x_1^2\end{bmatrix},\mathbf{D}_1=\begin{bmatrix}0.35\end{bmatrix},\mathbf{D}_2=\begin{bmatrix}0.26\end{bmatrix},\mathbf{D}_3=\begin{bmatrix}0.14\end{bmatrix},$$

$$\mathbf{D}_{1\omega}=\begin{bmatrix}1.24\end{bmatrix},\mathbf{D}_{2\omega}=\begin{bmatrix}1.19\end{bmatrix},\mathbf{D}_{3\omega}=\begin{bmatrix}1.05\end{bmatrix}.$$

Recalling the Lemma 1, it can be found that the open-loop PPFS is an positive system since $a_{irs}(x_1)$ is non-negative, for $i \in \{1,2,3\}$, $r \neq s$, and all of elements in the rest system matrices are non-negative. The disturbance signal is $\tilde{\mathbf{w}}(t)=\beta e^{-t}|cos(2t)|$, where $\beta = 1,2,3$, respectively.

In order to cut down the complexity of the SOF controller design, we choose 2 fuzzy rules for the SOF controller in this example. The MFs of the PPFS and the SOF controller are same as the ones in [18]. Through setting ε_1, $\varepsilon_2(x_1)$, $\varepsilon_3(x_1)$ and ε_4 as 0.001, the effectiveness of Theorem 1 is validated.

4.2. *Feasibility Analysis*

Fig. 1 and Figs. 2 to 4 show the time responses of x_1 and x_2 for the open-loop PPFS and the closed-loop PPFS, repectively. From Fig. 1, we can see that under zero initial condition, the open-loop PPFS is an unstable positive system because the time responses of x_1 and x_2 keep moving in the positive quadrant but do not converge to 0. From Figs. 2 to 4, it can be seen that the closed-loop system becomes an asymptotically stable and positive system since the time responses of x_1 and x_2 keep moving in the positive quadrant and converge to 0. Theorefore, it can be concluded that the designed SOF controller can guarantee the unstable PPFS to be stable and positive with satisfying the optimal L_1 performance on the basis of Theorem 1. Meanwhile, the obtained optimal performance is $\gamma = 1.959$, the feedback gains are $\mathbf{K}_1 = -3.0541$, $\mathbf{K}_2 = -3.0541$. Furthermore, based on the Corollary 1, the obtained optimal performance is $\gamma = 1.956$, the feedback gains are obtained $\mathbf{K}_1 = -3.0571$, $\mathbf{K}_2 = -3.0231$, $\mathbf{K}_3 = -3.0098$.

In order to investigate how the disturbance signal influence the stability of the closed-loop PPFS, we obtain different time responses of the states x_1 and x_2 when β is chosen as $\beta = 1,2,3$, respectively. In terms of the Figs. 2 to 4, we come to the conclusion that the stronger the disturbance signal, the slower the time response converges and the bigger the amplitude gets.

In order to demonstrate the superiority of the method proposed in this paper, we also compare it with another existing method in [19] and try to figure out the optimal performance index γ. Before making a comparison, we need to set $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}_3 = [0.36; 0.22]$ and keep the rest system matrices the same as the example above. That is because the method proposed in [19] requires the input matrices $\mathbf{B}_i$ to be assumed to be same for all i. The obtained performance indeces are $\gamma = 2.455$ and $\gamma = 1.945$ by using the method in [19] and the method in this paper, respectively, which indicates that the

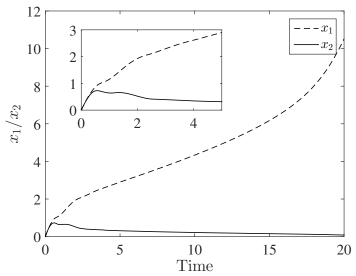

Figure 1. Time responses of the states x_1 and x_2 for the open-loop system.

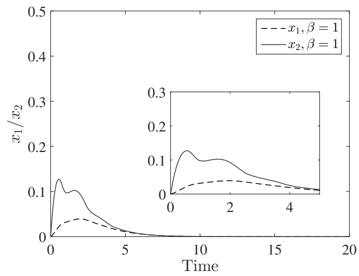

Figure 2. Time responses of the states x_1 and x_2 for the closed-loop system when $\beta = 1$.

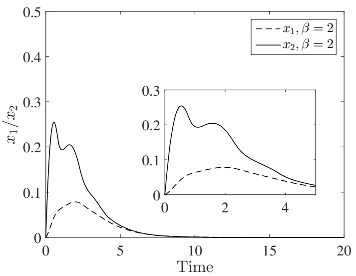

Figure 3. Time responses of the states x_1 and x_2 for the closed-loop system when $\beta = 2$.

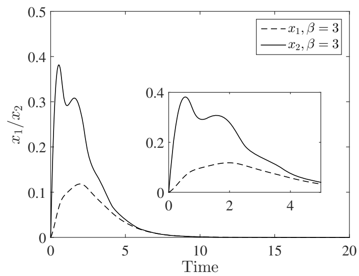

Figure 4. Time responses of the states x_1 and x_2 for the closed-loop system when $\beta = 3$.

method in this paper can provide better performance. In addition, another advantage of the method given in this paper over that one in [19] is that the manually chosen parameter is removed so that the conservativeness of the results can be reduced through eliminating the influence of human intervention.

5. Conclusion

In this paper, the positivity and stability analysis under L_1-induced performance for SOF PPFSs have been investigated. The SOF control approach has been employed to drive unstable PPFSs to be asymptotically stable and positive. Through introducing some extra constrain conditions, the tricky non-convex problem has been addressed. It is worth noting that this method has some merits than the method proposed in [19] since the manually chosen parameter has been removed which means the relaxation of the results has been improved by removing the human intervention factor. In addition, based on linear co-positive Lyapunov stability thoery, the SOS-based stability conditions have been derived. A simulation example has been given to illustrate the reliability and effectiveness of the proposed theorem.

Considering that many dynamical systems are with both discrete and continuous components in real systems [20,21], designing hybrid automata-based controllers for positive systems will be an interesting and challenging research topic for our future work.

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